Time discretizations for evolution problems

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Linear problem

Problem:

$$\begin{aligned} u' - \Delta u &= f, \quad \text{in } (0,T)\times\Omega \\ u &= 0, \quad \text{on } (0,T)\times\partial\Omega \\ u(0) &= u^0, \quad \text{in } \Omega \end{aligned}$$

Weak formulation: $u \in L^2(0,T,H_0^1(\Omega))$, $u' \in L^2(0,T,H^{-1}(\Omega))$

$$\begin{aligned} (u'(t),v) + (\nabla u(t),\nabla v) &= (f(t),v), \quad \forall v \in H_0^1(\Omega) \\ u(0) &= u^0 \end{aligned}$$

Finite element method

Let $\mathcal{T}_h = \{K\}$ be a conforming partition of Ω.
FEM space:

$$S_h = \{v \in H_0^1(\Omega) : v|_K \in P^p(K)\}$$

Let $R_h : H_0^1(\Omega) \to S_h$ be Ritz projection:

$$(\nabla(u - R_h u), \nabla v_h) = 0, \quad \forall v_h \in S_h$$

Approximation property of R_h:

$$\|u - R_h u\| \leq Ch^{p+1}|u|_{H^{p+1}(\Omega)}$$

We divide the error:

$$e^m = U^m - u^m = (U^m - R_h u^m) + (R_h u^m - u^m) = \xi^m + \eta^m$$

Euler method, θ-scheme

Notation: $t_m = m\tau$, $I_m = (t_{m-1}, t_m)$, $u^m = u(t_m)$
Approximation of derivative:

$$u'(t_{m-1}) \approx \frac{u^m - u^{m-1}}{\tau} \quad \text{or} \quad u'(t_m) \approx \frac{u^m - u^{m-1}}{\tau}$$

Explicit and implicit Euler method: $\{U^m\} \subset S_h$

$$\begin{aligned} (U^m - U^{m-1}, v_h) + \tau(\nabla U^{m-1}, \nabla v_h) &= \tau(f^{m-1}, v_h), \quad \forall v_h \in S_h \\ (U^m - U^{m-1}, v_h) + \tau(\nabla U^{m}, \nabla v_h) &= \tau(f^{m}, v_h), \quad \forall v_h \in S_h \end{aligned}$$

and its convex combination (θ-scheme): $\{U^m\} \subset S_h$

$$\begin{aligned} (U^m - U^{m-1}, v_h) + \tau(\nabla((1-\theta)U^{m-1} + \theta U^m), \nabla v_h) \\ = \tau((1-\theta)f^{m-1} + \theta f^m, v_h), \quad \forall v_h \in S_h \end{aligned}$$

Imlicit Euler analysis

Error equation:

$$(\xi^m - \xi^{m-1}, v_h) + \tau(\nabla \xi^m, \nabla v_h) = -(\eta^m - \eta^{m-1}, v_h) \\ -(u^m - u^{m-1} - \tau u'(t_m), v_h), \quad \forall v_h \in S_h$$

Setting $v_h = \xi^m$:

$$(\|\xi^m\| - \|\xi^{m-1}\|)\|\xi^m\| \le (\xi^m - \xi^{m-1}, \xi^m) + \tau\|\nabla \xi^m\|^2 \\ = -(\eta^m - \eta^{m-1}, \xi^m) - (u^m - u^{m-1} - \tau u'(t_m), \xi^m) \\ \le C\tau(h^{p+1} + \tau)\|\xi^m\|$$

After summation:

$$\|\xi^m\| \le \|\xi^0\| + Ct_m(h^{p+1} + \tau)$$

Explicit Euler analysis

Let $\{\varphi_i\} \subset S_h$ be eigenfunctions of $-\Delta$ satisfying:

$$(\nabla\varphi_i, \nabla v_h) = \lambda_i(\varphi_i, v_h), \quad \forall v_h \in S_h$$
$$(\nabla\varphi_i, \nabla\varphi_j) = \lambda_i\delta_{ij}, \qquad (\varphi_i, \varphi_j) = \delta_{ij}$$

Then the coordinates x_i^m of ξ^m satisfy:

$$x_i^m - x_i^{m-1} + \tau\lambda_i x_i^{m-1} = \tau RHS_i^m$$

$\Rightarrow$

$$\begin{aligned} x_i^m &= (1-\tau\lambda_i)x_i^{m-1} + \tau RHS_i^m \\ x_i^m &= (1-\tau\lambda_i)^m x_i^0 + \tau\sum_{j=0}^{m-1}(1-\tau\lambda_i)^{m-j-1} RHS_i^j \end{aligned}$$

Linear stability

Let us assume (Dalquist) equation:

$$y'(t) + \lambda y(t) = 0, \quad t \in (0, T),\ \lambda \in \mathbb{C}$$

Let us assume a numerical method providing $Y^{m-1} \to Y^m$. Let $R : \mathbb{C} \to \mathbb{C}$:

$$Y^m = R(\tau\lambda) Y^{m-1}.$$

We call R stability function and

$$S = \{z \in \mathbb{C} : |R(z)| \leq 1\}$$

stability domain.

Local error (Dahlquist eq.)

Let us assume (Dalquist) equation:

$$y'(t) + \lambda y(t) = 0, \quad t \in (0, T), \ \lambda \in \mathbb{C}$$

Let us assume a numerical method described by stability function R:

$$\begin{aligned} y^m &= e^{-\tau\lambda} y^{m-1} \\ Y^m &= R(\tau\lambda) Y^{m-1} \end{aligned}$$

Local error:

$$\mu^m = R(\tau\lambda) y^{m-1} - y^m = (R(\tau\lambda) - e^{-\tau\lambda}) y^{m-1}$$

Global error:

$$\begin{aligned}
e^m &= Y^m - y^m = R(\tau\lambda)Y^{m-1} - e^{-\tau\lambda}y^{m-1} \\
&= R(\tau\lambda)(Y^{m-1} - y^{m-1}) + (R(\tau\lambda) - e^{-\tau\lambda})y^{m-1} \\
&= R(\tau\lambda)e^{m-1} + \mu^m \\
&= \dots \\
&= R(\tau\lambda)^m e^0 + \sum_{j=0}^{m-1} R(\tau\lambda)^{m-j-1}\mu^j
\end{aligned}$$

Stability functions - Euler m.

Explicit Euler method stability function:

$$R(z) = 1 - z, \qquad \lambda \geq 0 \Rightarrow \left(|R(\tau\lambda)| \leq 1 \Leftrightarrow \tau \leq \frac{2}{\lambda} \right)$$

Local error of explicit Euler m.:

$$1 - z - e^{-z} = O(z^2)$$

Implicit Euler method stability function:

$$R(z) = \frac{1}{1+z}, \qquad |R(\tau\lambda)| \leq 1 \Leftarrow \mathrm{Re}\lambda \geq 0$$

Local error of explicit Euler m.:

$$\frac{1}{1+z} - e^{-z} = O(z^2)$$

Eigenvalues of $-\Delta$

Example: $d = 1$, $\Omega = (0, 1)$, $h = 1/N$, $p = 1$:

$$\lambda_N = 12N^2 \frac{1 - cos\left(\pi \frac{N-1}{N}\right)}{4 + 2cos\left(\pi \frac{N-1}{N}\right)} \approx 12N^2 = \frac{12}{h^2}$$

Stability condition for explicit Euler method:

$$\tau \leq \frac{2}{\lambda_N} \approx \frac{h^2}{6}$$

Crank-Nicolson m. ($\theta = 1/2$)

Stability function:

$$R(z) = \frac{1 - z/2}{1 + z/2}, \qquad |R(\tau\lambda)| \leq 1 \Leftrightarrow \mathrm{Re}\lambda \geq 0$$

Local error:

$$\frac{1 - z/2}{1 + z/2} - e^{-z} = O(z^3)$$

Numerical experiment - stability

Equation:

$$u' - \Delta u = f$$

Exact solution:

$$u(t,x) = 4\frac{e^{10t}-1}{e^{10}-1}x(1-x)$$

$1/h$	$1/\tau$	explicit Euler	implicit Euler	Crank-Nicolson
100	5	$3.09E+7$	$2.92E-1$	$9.95E-2$
100	20	$1.68E+60$	$8.76E-2$	$7.51E-3$
100	40	$4.02E+123$	$4.49E-2$	$1.88E-3$
100	80	-	$2.27E-2$	$4.50E-4$
100	160	-	$1.14E-2$	$9.82E-5$

Linear multi-step methods:

$$\sum_{i=0}^{k} \alpha_i (U^{m-i}, v_h) + \tau \beta_i (\nabla U^{m-i}, \nabla v_h) = \tau \sum_{i=0}^{k} \beta_i (f^{m-i}, v_h), \quad \forall v_h \in S_h$$

Assumptions: $k \geq 1$, $\alpha_0 > 0$, $|\alpha_k| + |\beta_k| > 0$

Local order:

$$\sum_{i=0}^{k} \alpha_i = 0, \qquad \sum_{i=0}^{k} \alpha_i i^s = s \sum_{i=0}^{k} \beta_i i^{s-1}, \quad s = 1, \ldots, q$$

Multi-step methods, stability

Multi-step method for Dahlquist eq.:

$$\sum_{i=0}^{k} \alpha_i Y^{m-i} + \tau\lambda \sum_{i=0}^{k} \beta_i Y^{m-i} = \sum_{i=0}^{k} (\alpha_i + \tau\lambda\beta_i) Y^{m-i} = 0$$

Characteristic polynomial:

$$\rho(x, z) = \sum_{i=0}^{k} (\alpha_i + z\beta_i) x^{k-i}$$

Multi-step method is stable, iff the roots $x_i(z)$ of $\rho(x, z)$ are less or equal to 1 in modulus and those with modulus 1 has multiplicity 1.

Adams methods

Adams methods:

$$(U^m - U^{m-1}, v_h) + \tau \sum_{i=0}^{k} \beta_i (\nabla U^{m-i}, \nabla v_h) = \tau \sum_{i=0}^{k} \beta_i (f^{m-i}, v_h) \quad \forall v_h \in S_h$$

Local order:
explicit (Adams-Bashforth) methods ... k
implicit (Adams-Moulton) methods ... $k+1$

Stability, Adams methods

Roots of characteristic polynomials (Adams m.):

k	explicit	implicit
1	$1 - z$	$\frac{1-z/2}{1+z/2}$
2	$\frac{2-3z\pm\sqrt{9z^2-4z+4}}{4}$	$\frac{1-2z/3\pm\sqrt{7z^2/12-z+1}}{2+5z/6}$

Restrictions (on the real line):

k	explicit	implicit
1	$z \leq 2$	-
2	$z \leq 1$	$z \leq 6$
3	$z \leq 0.545$	$z \leq 3$
4	$z \leq 0.3$	$z \leq 1.837$

BDF

BDF = Backward Differentiation Formula:

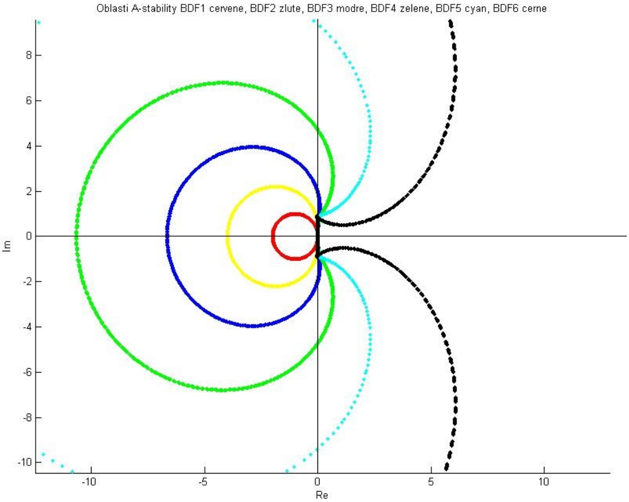
$$\sum_{i=0}^{k} \alpha_i (U^{m-i}, v_h) + \tau(\nabla U^m, \nabla v_h) = \tau(f^m, v_h), \quad \forall v_h \in S_h$$

Coefficients:

$$\alpha_0 = \sum_{i=0}^{k} \frac{1}{i}, \qquad \alpha_i = \frac{(-1)^i}{i} \binom{k}{i}, \quad i = 1, \ldots, k$$

BDF have local order k.

Stability, BDF



BDF $k = 1, \ldots, 6$, analysis

Let

$$(Au, v) = (\nabla u, \nabla v), \quad \forall u, v \in H_0^1(\Omega)$$

We define the sequence of operators $\{\gamma_j\}_{j=0}^{\infty}$

$$(\tau A + \mathrm{Id} \sum_{i=0}^{k} \alpha_i z^{k-i})^{-1} = \sum_{j=0}^{\infty} \gamma_j z^j, \quad \forall z \geq 0$$

Then

$$\begin{aligned} \|\gamma_j\| &\leq C, \quad \forall j = 0, 1, \ldots \\ \tau \sum_{j=0}^{\infty} (\nabla(\gamma_j v), \nabla(\gamma_j v)) &\leq C\|v\|^2, \quad \forall v \in H_0^1(\Omega) \end{aligned}$$

BDF $k = 1, \dots, 6$, analysis

Error equation:

$$\left(\sum_{i=0}^{k} \alpha_i \xi^{m-i} + \tau A \xi^m, v_h \right) = \tau (RHS^m, v_h)$$

Setting $v_h = \gamma_{n-m} \xi^n$, summing $m = k, \dots, n$:

$$\|\xi^n\|^2 = -\left(\sum_{s=0}^{k-1} \sum_{i=0}^{s} \alpha_{k-i} \gamma_{n-s-i} \xi^s, \xi^n \right) + \tau \sum_{m=k}^{n} (RHS^m, \gamma_{n-m} \xi^n)$$

BDF, variable step size

Coefficients of BDF depend on $\tau_1, \ldots$:

$$\sum_{i=0}^{k} \alpha_i(\tau_1, \ldots, \tau_k)(U^{m-i}, v_h) + \tau_k(\nabla U^m, \nabla v_h) = \tau_k(f^m, v_h)$$

$k = 2$: steps: τ_1, τ_2, stable for $0 < \tau_2 < (1 + \sqrt{2})\tau_1$

$$\alpha_0 = \frac{\tau_1 + 2\tau_2}{\tau_1 + \tau_2}, \quad \alpha_1 = -\frac{\tau_1 + \tau_2}{\tau_1}, \quad \alpha_2 = \frac{\tau_2^2}{\tau_1(\tau_1 + \tau_2)}$$

$k = 3$: τ_1, τ_2, τ_3 ... complicated

Runge-Kutta methods:

$$(g_i, v_h) + \tau \sum_{j=1}^{k} a_{i,j}(\nabla g_j, \nabla v_h) = (U^{m-1}, v_h)$$

$$+\tau \sum_{j=1}^{k} a_{i,j}(f(t_{m-1} + \tau c_j), v_h), \quad \forall v_h \in S_h, \ i = 1, \dots, k$$

$$(U^m - U^{m-1}, v_h) + \tau \sum_{i=1}^{k} b_i(\nabla g_i, \nabla v_h)$$

$$= \tau \sum_{i=1}^{k} \beta_i(f(t_{m-1} + \tau c_i), v_h), \quad \forall v_h \in S_h$$

Runge–Kutta methods, coefficients

Coefficients of Runge–Kutta method:

$$\begin{array}{c|ccc} c_1 & a_{1,1} & \cdots & a_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ c_k & a_{k,1} & \cdots & a_{k,k} \\ \hline & b_1 & \cdots & b_k \end{array} = \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

Runge-Kutta m., stability

Runge-Kutta method for Dahlquist eq.:

$$
\begin{aligned}
g_i + \tau\lambda \sum_{j=1}^{k} a_{i,j} g_j &= Y^{m-1}, \quad i = 1, \dots, k \\
Y^m + \tau\lambda \sum_{i=1}^{k} b_i g_i &= Y^{m-1}
\end{aligned}
$$

Matrix formulation:

$$
\begin{aligned}
g + \tau\lambda A g &= e Y^{m-1} \\
Y^m + \tau\lambda b^T g &= Y^{m-1}
\end{aligned}
$$

Stability function:

$$
\begin{aligned}
Y^m &= (1 - \tau\lambda b^T (I + \tau\lambda A)^{-1} e) Y^{m-1} \\
R(z) &= 1 - z b^T (I + zA)^{-1} e
\end{aligned}
$$

Runge-Kutta m., stability

Matrix formulation (of R.-K.):

$$\left(\begin{array}{c|c} I + zA & 0 \\ \hline zb^T & 1 \end{array}\right) \left(\begin{array}{c} g \\ \hline Y^m \end{array}\right) = \left(\begin{array}{c} e \\ \hline 1 \end{array}\right) Y^{m-1}$$

Cramer's rule:

$$\begin{aligned} Y^m &= \frac{\det(I + zA - zeb^T)}{\det(I + zA)} Y^{m-1} \\ R(z) &= \frac{\det(I + zA - zeb^T)}{\det(I + zA)} \end{aligned}$$

Padé approximations

(i,j)-Padé approximations to e^{-z}:

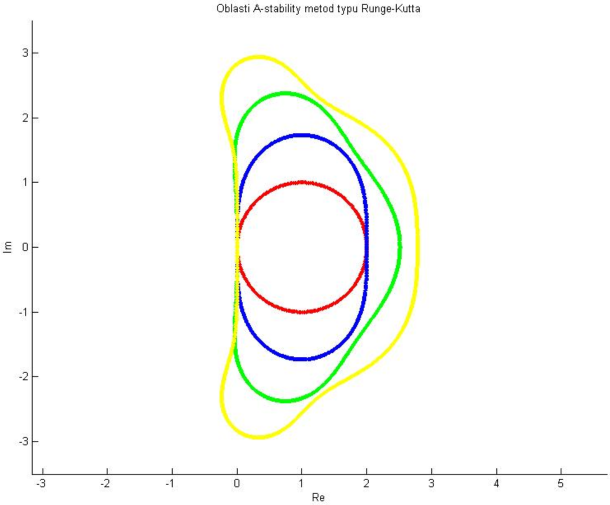
1	$1-z$	$1-z+z^2/2$
$\frac{1}{1+z}$	$\frac{1-z/2}{1+z/2}$	$\frac{1-2z/3+z^2/6}{1+z/3}$
$\frac{1}{1+z+z^2/2}$	$\frac{1-z/3}{1+2z/3+z^2/6}$	$\frac{1-z/2+z^2/12}{1+z/2+z^2/12}$
$\frac{1}{1+z+z^2/2+z^3/6}$	$\frac{1-z/4}{1+3z/4+z^2/4+z^3/24}$	$\frac{1-2z/5+z^2/20}{1+3z/5+3z^2/20+z^3/60}$

Local order of R.-K.:

$$e^{-z} - R_{i,j}(z) = O(z^{i+j+1})$$

i,j-Padé approximation is A-stable, iff $j-2 \leq i \leq j$.

Stability, ERK



Collocation methods

Collocation polynomial on $P^k(I_m, S_h)$:

$$\begin{aligned} P(t_{m-1}) &= U^{m-1} \\ (P', v_h) + (\nabla P, \nabla v_h) &= (f, v_h) \\ & \quad \forall v_h \in S_h,\ t = t_{m-1} + \tau c_i,\ i = 1, \dots, k \end{aligned}$$

Then

$$U^m = P(t_m)$$

Collocation m. $\subset$ Runge-Kutta m.

Let $\ell_j \in P^{k-1}$, $\ell_j(c_i) = \delta_{ij}$. Then collocation m. with collocation points $t_{m-1} + \tau c_i$ is equivalent to Runge-Kutta m. with coefficients c_i,

$$a_{i,j} = \int_0^{c_i} \ell_j(t)dt, \qquad b_i = \int_0^1 \ell_i(t)dt$$

Gauss and Radau collocation m.

Let c_i are the nodes of Gauss quadrature on $[0, 1]$. Then the corresponding collocation method stability function is (k, k)-Padé approximation.
Let c_i are the nodes of (right) Radau quadrature ($c_k = 1$) on $[0, 1]$. Then the corresponding collocation method stability function is $(k - 1, k)$-Padé approximation. (Radau IIA Runge-Kutta m.)

Weak formulation

Spaces:

$$\begin{aligned} X &= L^2(0, T, H_0^1(\Omega)) \\ Y &= \{v \in X : v' \in L^2(0, T, H^{-1}(\Omega))\} \\ Y_0 &= \{v \in Y : v(0) = u^0\} \end{aligned}$$

Weak formulation: $u \in Y_0$

$$\int_0^T (u', v) + (\nabla u, \nabla v) dt = \int_0^T (f, v) dt, \quad \forall v \in X$$

Continuous Galerkin method

Discrete spaces:

$$
\begin{aligned}
Y_h^\tau &= \{v \in Y : v|_{I_m \times K} \in P^k(I_m) \otimes P^p(K), v(0) = \Pi^{L^2} u^0\} \\
X_h^\tau &= \{v \in X : v|_{I_m \times K} \in P^{k-1}(I_m) \otimes P^p(K)\}
\end{aligned}
$$

Continuous Galerkin method: $U \in Y_h^\tau$

$$
\int_0^T (U', v) + (\nabla U, \nabla v) dt = \int_0^T (f, v) dt, \quad \forall v \in X_h^\tau
$$

Discontinuous Galerkin method

Discrete space:

$$X_h^\tau = \{v \in X : v|_{I_m \times K} \in P^{k-1}(I_m) \otimes P^p(K)\}$$

Discontinuous Galerkin method: $U \in X_h^\tau$

$$\int_{I_m} (U', v) + (\nabla U, \nabla v) dt + ([U]_{m-1}, v_+^{m-1}) = \int_{I_m} (f, v) dt, \quad \forall v \in X_h^\tau, \forall m$$

Galerkin m. $\subset$ collocation m.

Let c_i be Gauss quadrature nodes on $[0,1]$ and P be the corresponding collocation solution on I_m. Let U be the solution of continuous Galerkin m. with Gauss quadrature. Then

$$P = U$$

Let c_i be (right) Radau quadrature nodes on $[0,1]$ and P be the corresponding collocation solution on I_m. Let U be the solution of discontinuous Galerkin m. with Radau quadrature. Let $r_m \in P^k$, $r_m(t_{m-1} + \tau c_i) = 0$, $r_m(t_{m-1}) = 1$. Then

$$P = U - [U]_{m-1} r_m$$

Local error (Dahlquist eq.):

$$\mu^m = (R(\tau\lambda) - e^{-z})y^{m-1} = C(\tau\lambda)^{q+1}y^{m-1} = C\tau^{q+1}y^{(q+1)}$$

Global error estimates for Dahlquist eq.:

$$\begin{aligned} e^m = Y^m - y^m &= R(\tau\lambda)e^{m-1} + (R(\tau\lambda) - e^{-z})y^m \\ &= \ldots = R(\tau\lambda)^m e^0 + \sum_{j=0}^{m-1} R(\tau\lambda)^{m-j-1}\mu^m \end{aligned}$$

Radau IIA Runge-Kutta m. has order (classical order) $2k-1$.
Often reduced order (stiff order) k is observed.

Radau IIA R.-K. m., order reduction

Extension of Dahlquist eq. to general operator eq.:

$$y' + By = 0$$

Since

$$\|R(B)\| \leq \sup_{\lambda \in \sigma(B)} |R(\lambda)|$$

the analysis is completely the same.
Due to complicated character of Runge-Kutta m. equations:

$$y' + By = 0$$
$$y' + By = f$$

orders of Runge-Kutta m. behave differently.
Additional condition:

$$y^{(s)} \in \mathrm{Dom}(B^{2k-s}), \quad s = k+1, \dots, 2k$$

Order reduction, example

Exact solution:

$$y(t) = cos\left(\frac{\pi}{4} + 2\pi t\right)$$

Equation:

$$y' + \lambda y = \lambda\varphi(t) + \varphi'(t),$$

$\lambda = 10, 1000, 100000$

Order reduction, example, DG $k = 3$

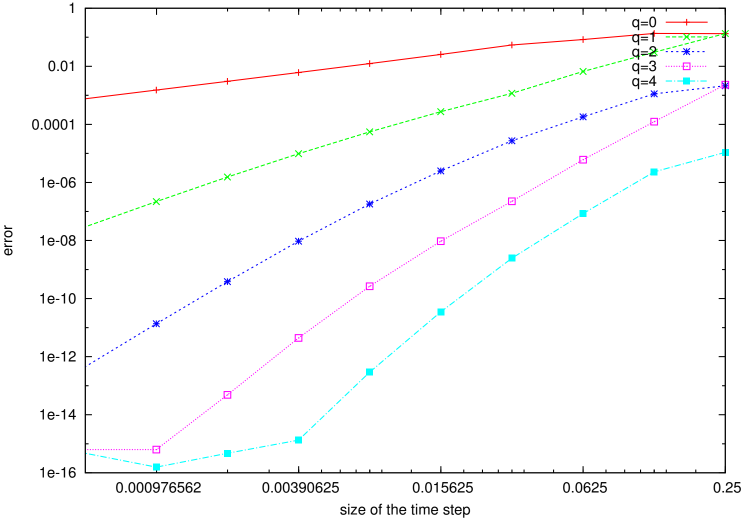
$\lambda = 10$:

τ	$\lvert e(t_m)\rvert$	EOC	$\lvert e(t_r)\rvert$	EOC	$\lvert e(t_i)\rvert$	EOC
2.50E-01	4.34E-05	–	2.31E-03	–	2.38E-03	–
1.25E-01	1.10E-05	2.0	5.86E-05	5.3	1.92E-03	0.3
6.25E-02	4.25E-07	4.7	5.73E-06	3.4	3.72E-04	2.4
3.12E-02	2.41E-08	4.1	6.59E-07	3.1	6.59E-05	2.5
1.56E-02	5.87E-10	5.4	3.93E-08	4.1	8.02E-06	3.0
7.80E-03	1.56E-11	5.2	2.39E-09	4.0	9.87E-07	3.0
3.90E-03	4.43E-13	5.1	1.48E-10	4.0	1.22E-07	3.0

$\lambda = 100000$:

τ	$\lvert e(t_m)\rvert$	EOC	$\lvert e(t_r)\rvert$	EOC	$\lvert e(t_i)\rvert$	EOC
2.50E-01	2.46E-03	–	1.05E-03	–	2.46E-03	–
1.25E-01	1.23E-03	1.0	3.92E-04	1.4	1.23E-03	1.0
6.25E-02	2.06E-04	2.6	7.33E-05	2.4	2.06E-04	2.6
3.12E-02	3.43E-05	2.6	1.27E-05	2.5	3.43E-05	2.6
1.56E-02	4.08E-06	3.1	1.52E-06	3.1	4.08E-06	3.1
7.80E-03	4.95E-07	3.0	1.87E-07	3.0	4.95E-07	3.0
3.90E-03	6.06E-08	3.0	2.32E-08	3.0	6.06E-08	3.0
1.95E-03	7.40E-09	3.0	2.91E-09	3.0	7.41E-09	3.0
9.75E-04	8.97E-10	3.0	3.72E-10	3.0	9.08E-10	3.0
4.87E-04	1.04E-10	3.1	4.74E-11	3.0	1.09E-10	3.0

Order reduction, example, DG $\lambda = 1000$



Nonlinear problem

Nonlinear problem:

$$u' + A(u) = f$$

One-sided Lipschitz condition:

$$(A(u) - A(v), u - v) \geq \nu \|u - v\|^2$$

Error equation:

$$(\xi^m - \xi^{m-1}, v_h) + \tau(A(U^m) - A(\Pi u^m), v_h) = \tau(RHS^m, v_h)$$

Setting $v_h = \xi^m$:

$$(\|\xi^m\| - \|\xi^{m-1}\|)\|\xi^m\| + \tau\nu\|\xi^m\|^2 \leq \tau\|RHS^m\| \, \|\xi^m\|$$

From Gronwall lemma:

$$\|\xi^m\| \leq e^{-t_m\nu}(\|\xi^0\| + T \max_j \|RHS^j\|)$$

Time dependent domains, ALE

Assume time dependent problem:

$$u' + A_t u = f, \quad \text{in } (0, T) \times \Omega_t$$

Ale mapping:

$$\mathcal{A}_t : \Omega_0 \to \Omega_t$$

Mesh velocity:

$$\omega(t, x) = \frac{\partial \mathcal{A}_t}{\partial t}(\mathcal{A}_t^{-1}(x))$$

Ale derivative:

$$D_t u = u' + \omega \cdot \nabla u$$

ALE formulation, discretization

Ale formulation:

$$D_t u + A_t u - \omega \cdot \nabla u = f$$

Discretization (implicit Euler m.):

$$U^m - U^{m-1} \circ \mathcal{A}^{-1}_{t_{m-1},t_m} + \tau A_t U^m - \tau \omega \cdot \nabla U^m = \tau f^m$$

Exponential integrators

Assume equation:

$$u' + Au = f$$

Variation of constants:

$$u^m = e^{-\tau A} u^{m-1} + \int_{I_m} e^{-(t_m - s)A} f(s) ds$$

Approximation of f by polynomial $\tilde{f}$:

$$U^m = e^{-\tau A} U^{m-1} + \int_{I_m} e^{-(t_m - s)A} \tilde{f}(s) ds$$

We need to be able to compute:

$$e^{-\tau A}, \qquad \int_{I_m} e^{-(t_m - s)A} s^j ds$$

Global error:

$$
\begin{aligned}
e^m &= U^m - u^m = e^{-\tau A} + \int_{I_m} e^{-(t_m - s)A}(\tilde{f}(s) - f(s))ds \\
&= e^{-t_m A} e^0 + \sum_{j=1}^{m} \int_{I_j} e^{-(t_m - s)A}(\tilde{f}(s) - f(s))ds
\end{aligned}
$$

Thank you for your attention.