DG method for the numerical pricing of path-dependent multi-asset options

Jiří Hozman [1] and Tomáš Tichý [2]

[1]Technical University of Liberec
Faculty of Science, Humanities and Education

[2]VŠB-TU Ostrava
Faculty of Economics

Programy a algoritmy numerické matematiky 18
Janov nad Nisou,
June 19 – 24, 2016

Outline

1. Motivation
   - Financial background
   - Basket options
   - Asian options
2. Path-dependent multi-asset options
3. Asian two-asset option with floating strike
   - PDE formulation
   - Weak formulation
   - DG discretization
   - Numerical experiments
4. Asian two-asset option with fixed strike
   - Similar approach
   - Numerical experiments

1 Motivation

2 Path-dependent multi-asset options

3 Asian two-asset option with floating strike

4 Asian two-asset option with fixed strike

Objects of our interest

- to develop efficient, accurate and robust numerical scheme for the simulation of option pricing problem within DG technique,
- to optimize particular steps of the approximation to obtain correct results attainable in a reasonable time,
- to be robust with respect to various types of options as well as market conditions,
- to analyze the impact of various approaches
- to examine the computational error and demandingness

Basic mechanism of options

- **option** - a special case of financial derivatives, contract between two parties about trading the asset at a certain future time

$$\textbf{writer} \xrightarrow{\text{sells}} \begin{array}{c}\textbf{premium}\\ \text{(market price)}\end{array} \xrightarrow{\text{purchases}} \textbf{holder}$$

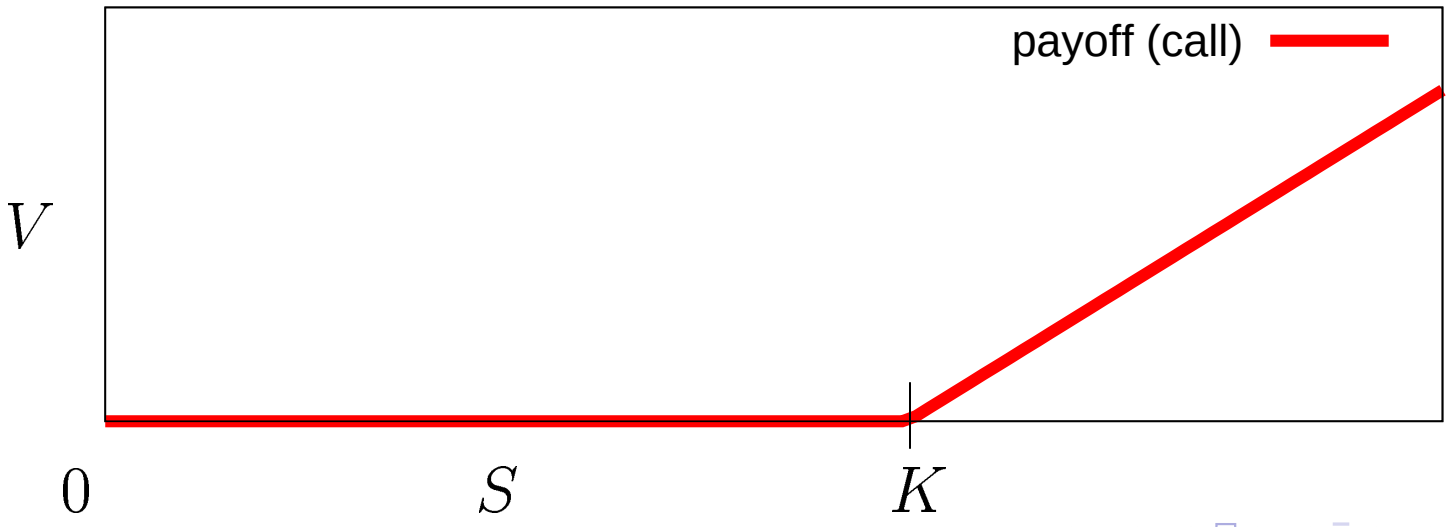
- **option** has a limited life time: maturity or expiration time T
- we distinguish:
  - **call** - gives the holder the right to *buy* the underlying for an agreed price K (strike) by the date T
  - **put** - gives the holder the right to *sell* the underlying for an agreed price K by the date T

What is the price of such a contract?

Option value

- at expiration T the value of option $V(S,t)$ is clearly given by **payoff** function

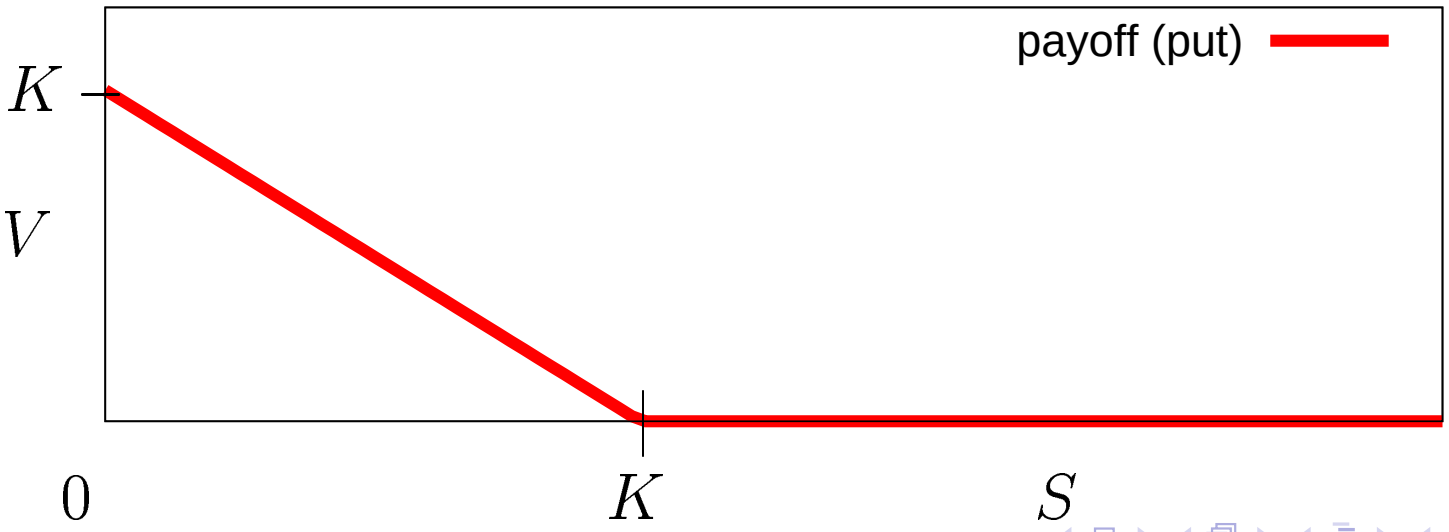
$$V(S,0) = \begin{cases} \max(S-K,0), & \text{for a call} \\ \max(K-S,0), & \text{for a put} \end{cases}$$



Option value

- at expiration T the value of option $V(S,t)$ is clearly given by **payoff** function

$$V(S,0) = \begin{cases} \max(S-K,0), & \text{for a call} \\ \max(K-S,0), & \text{for a put} \end{cases}$$



Option value

- at expiration T the value of option $V(S,t)$ is clearly given by **payoff** function

$$V(S,0) = \begin{cases} \max(S-K,0), & \text{for a call} \\ \max(K-S,0), & \text{for a put} \end{cases}$$

But what about now?
How much would you pay for such an option?
How to calculate a fair value of the option?

Option pricing models

Option value

- at expiration T the value of option $V(S,t)$ is clearly given by **payoff** function

$$V(S,0) = \begin{cases} \max(S-K,0), & \text{for a call} \\ \max(K-S,0), & \text{for a put} \end{cases}$$

But what about now?
How much would you pay for such an option?
How to calculate a fair value of the option?

Option pricing models

Introduction to option pricing

- a financial asset with tendency μ and volatility σ

$$dS_t = S_t(\mu dt + \sigma dW_t), S_0 \text{ known} \tag{1}$$

- μ is drift of S_t, $\mu = r$, where r is risk-free interest rate
- W_t is a standard Brownian motion (BM)
- for constant σ and r we have solution of SDE (1) as

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} \quad \text{(geometric BM)}$$

- option value at t is the expected profit at T discounted to t:

$$\begin{aligned} C_t &= e^{-r(T-t)}\,\mathbb{E}(\max(S_T - K, 0)) \quad \text{(call)} \\ P_t &= e^{-r(T-t)}\,\mathbb{E}(\max(K - S_T, 0)) \quad \text{(put)} \end{aligned}$$

- due to the put call parity we can either compute calls or puts

$$C_t - P_t = S_t - Ke^{-r(T-t)}$$

Numerical techniques for option pricing

1. Monte-Carlo simulations

$$S_{k+1}^n - S_k^n = S_k^n(\mu\delta t + \sigma\sqrt{\delta t}\mathcal{N}(0,1)),\ S_0^n = S_0$$

$$C_t = e^{-r(T-t)}\frac{1}{N}\sum_{n=1}^{N}(\max(S_T^n - K, 0)),$$

   where $\mathcal{N}(0,1)$ is approximated by random function

2. tree methods (Cox, Ross and Rubinstein binomial model)
3. PDE approach within Ito calculus

$$df(t, S_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS_t dS_t)$$

$\longrightarrow$ **Black-Scholes (BS) equation**

Monte-Carlo vs. PDE approach

Monte-Carlo simulation

- near to the modelisation,
- easy to parallelize,
- sensitivity parameters (Greeks) using Malliavin calculus,
- calibration is very hard

PDE aproach

- nearer to the analytical formulas,
- can be parallelized,
- sensitivity parameters (Greeks) are easy to evaluate,
- calibration is feasible

Derivation of BS equation

We make the following assumptions:

- the assets prices follow the lognormal random walk,
- the risk-free interest rate r, the volatility σ are constant,
- no transactions costs associated with hedging a portfolio,
- no arbitrage possibilities,
- the underlying assets pay no dividends,
- trading of the underlying assets can take place continuously,
- short selling is permitted and the assets are divisible.

We construct a portfolio of option and Δ shares of the underlying

$$\Pi = V(S,t) - \Delta S$$

where V represents a long position (it is owned) and ΔS a short position (it is owed)

Derivation of BS equation

(A) the change in our portfolio from t to $t + dt$ is completely riskless (riskless portfolio), described by ODE,

$$d\Pi = r\Pi dt$$

(B) equivalently, the change in our portfolio is also equal to

$$d\Pi = dV - \Delta dS$$

and using Ito lemma with $(dSdS) \backsimeq \sigma^2 S^2 dt$ we have

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt$$

i.e., the portfolio changes by

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \left(\frac{\partial V}{\partial S} - \Delta \right) dS$$

Derivation of BS equation

- randomness is eliminated when $\Delta = \frac{\partial V}{\partial S}$ in our portfolio (delta hedging, dynamic hedging strategy), i.e.

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

- putting (A) and (B) together we obtain

$$r\Pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

finally, from $\Pi = V - \Delta S = V - \frac{\partial V}{\partial S} S$ we get the famous Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Black-Scholes (BS) equation

- derivation of BS equation [Black, Scholes, 1973],
- a Nobel prize was awarded in 1997

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- $V = V(S, t; T, K, r, \sigma)$ - value of option
- t, $0 \leq t \leq T$ - current time, $T - t$ - time time to expiration
- S - price of stock at time t, i.e. $S = S(t)$, $S \in (0; +\infty)$
- r, $r > 0$ - risk-free interest rate
- σ, $\sigma > 0$ - volatility of price S
- K - strike price

- backward linear parabolic differential equation

Black-Scholes formula

- by transforming the Black-Scholes PDE into the heat equation with reversal time
- Black-Scholes formula

$$C_t = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$
$$P_t = -S_t\Phi(-d_1) + Ke^{-r(T-t)}\Phi(-d_2)$$

where

$$d_1 = \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

and Φ denotes the cumulative distribution function of the standard normal distribution

BS equation – numerical experiments

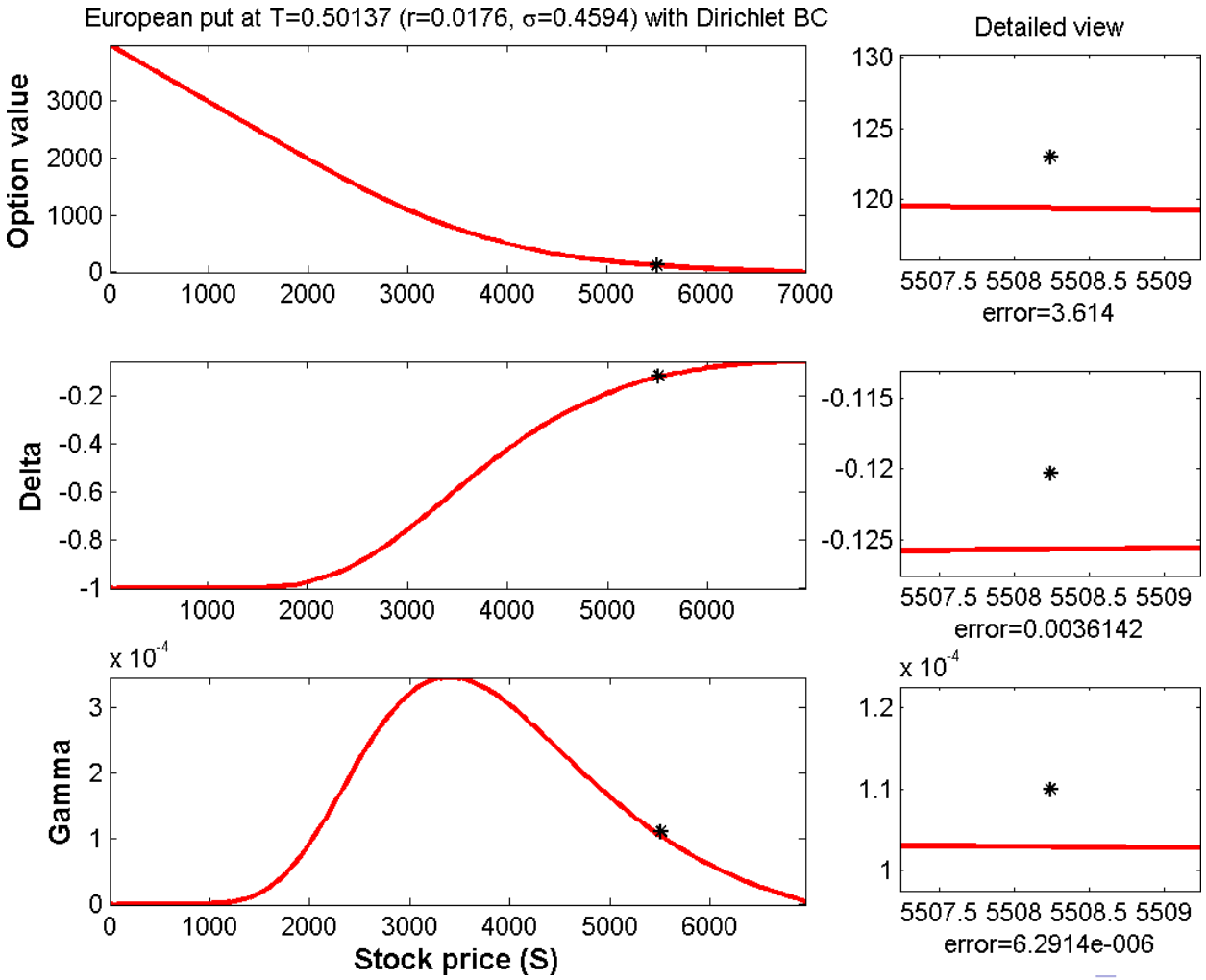
- real data of the DAX on 15SEPT2011 with implied volatilities,
- expiration date $T = 0.50137$ (half year),
- risk-free interest rate $r = 0.0176$,
- uniform space-time grid with constant mesh step h and constant time step τ,
- DG piecewise P_3 (cubic) approximations,
- sensitivity measures (Greeks):

$$\Delta = \frac{\partial u}{\partial x} \quad \text{and} \quad \Gamma = \frac{\partial^2 u}{\partial x^2}$$

- we investigate the behaviour of approximation values of option and Greeks w.r.t. the choice of b.c.
- reference DAX data at underlying's price $x = 5508.238$ (star-symbol in graphs)

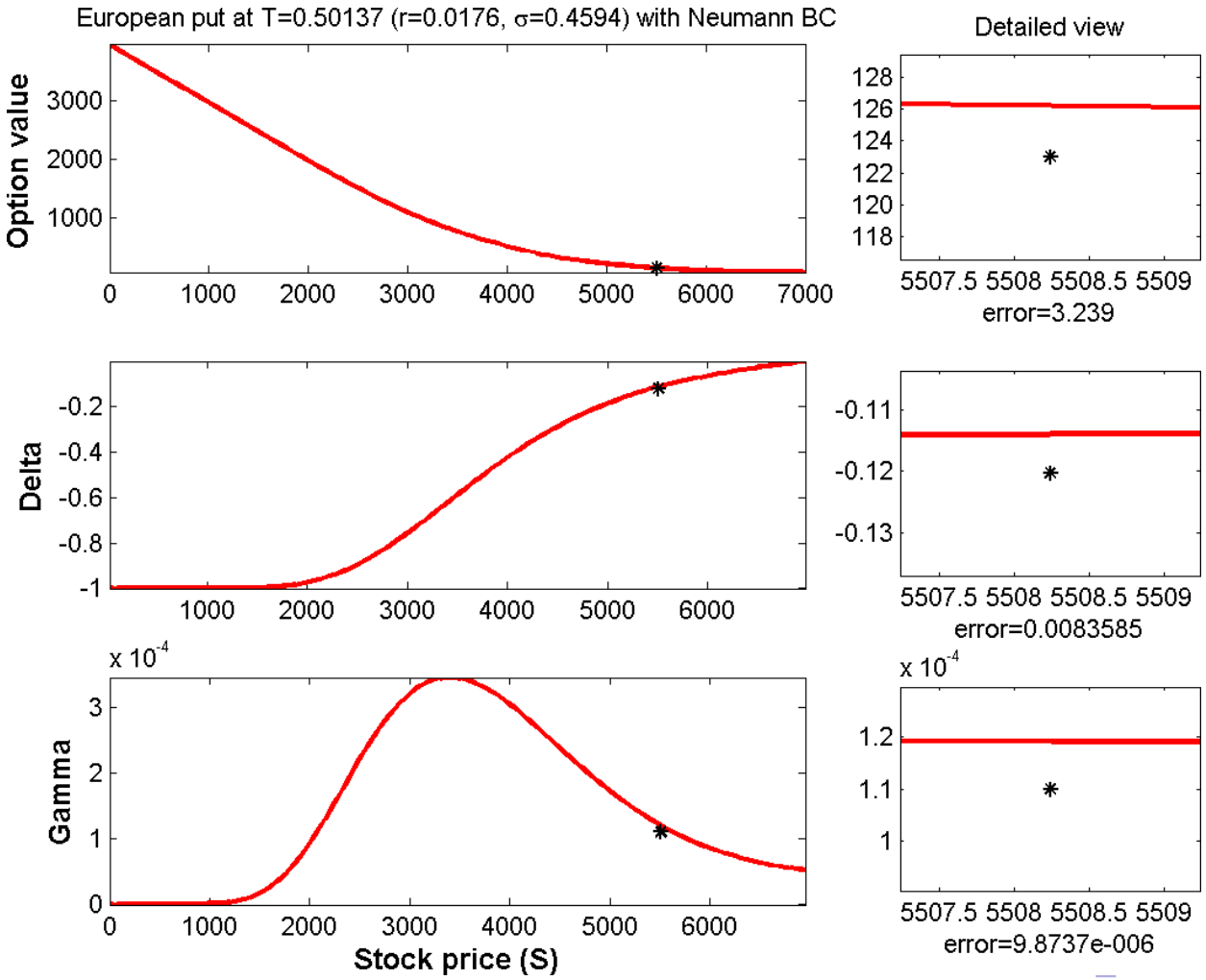
Put options - comparison at expiration date

- opt. val. and Greeks (Dirichlet b.c., $K = 4000$), ref. val.($*$)



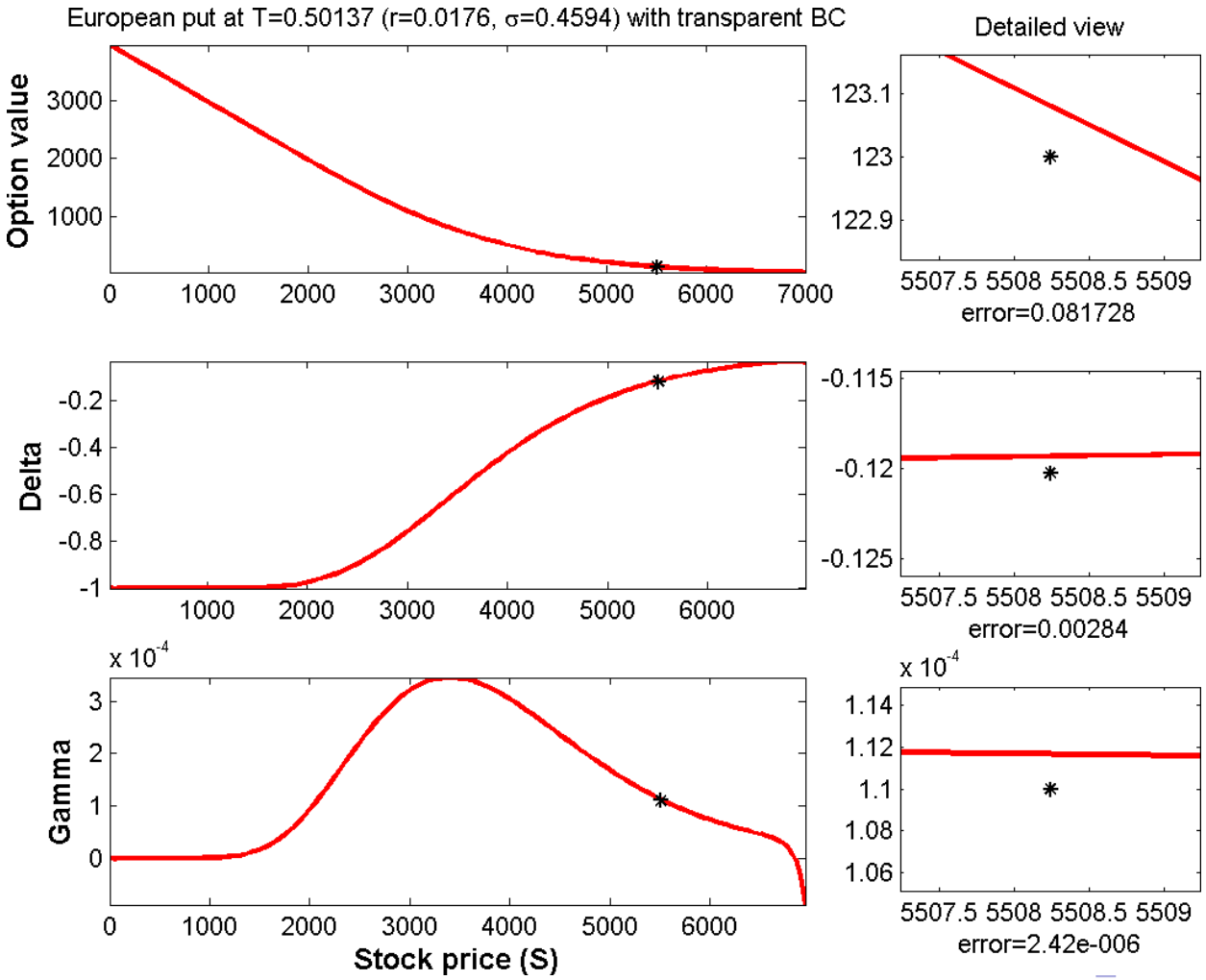
Put options - comparison at expiration date

- opt. val. and Greeks (Neumann b.c., $K = 4000$), ref. val.($*$)



Put options - comparison at expiration date

- opt. val. and Greeks (transparent b.c., $K = 4000$), ref. val.($*$)



Limitations and generalizations of the BS model

- idealized assumptions are rarely valid in the real world
  - normal distribution of log-returns
  - constant model parameters
- important market features are not captured by BSM
  - volatility clustering and leverage effect
  - volatility smile
- generalizations of BS framework
  - assumptions of non-constant deterministic (e.g. local volatility) or stochastic parameters (e.g. Heston model)
  - other models of the price movements instead of diffusion one (in BS model) such as jump-diffusion model (leading to PIDE) or pure jump model (e.g. VG model)

Classification of options

- from the holder's point of view
  - calls - a right to buy
  - puts - a right to sell
- according to the option style
  - European - may only be exercised on expiry
  - American - may be exercised on any trading day on or before expiration
- according to the relation between asset price S and strike K
  - out of the money (OTM): $S < K$ (call) and $S > K$ (put)
  - at the money (ATM): $S = K$ (call and put)
  - in the money (ITM): $S > K$ (call) and $S < K$ (put)
- standard (vanilla) option vs. non-standard (exotic) options
  - discontinuous payoff function (e.g. barrier)
  - depend on a basket of several underlying assets (e.g. basket)
  - path-dependent options (e.g. Asian)

Basket options

- generally, dimension $d \geq 2 \Rightarrow$ multi-asset options
- option depending on a basket of two underlying assets

$$\left.\begin{array}{l}\text{price of } 1^{st} \text{ asset } S_1 \\ \text{price of } 2^{nd} \text{ asset } S_2\end{array}\right\} \Rightarrow \textbf{value of basket option } V(S_1, S_2, t),$$

- movement of stock prices is driven by SDEs

$$\begin{aligned} dS_1(t) &= rS_1(t)dt + \sigma_1 S_1(t)dW_1(t) \\ dS_2(t) &= rS_2(t)dt + \sigma_2 S_2(t)dW_2(t) \end{aligned}$$

with correlated Wiener processes, i.e., $dW_1(t)dW_1(t) = \rho dt$,

- $S_k(t)$ - prices of stock at time t, $0 \leq S_k < \infty, k = 1, 2$,
- r, $r > 0$ - risk-free interest rate,
- σ_k, $\sigma_k > 0$ - volatility of the k-th price S_k, $k = 1, 2$

2D Black-Scholes equation

- analogous approach to 1D BS model with risk-free portfolio

$$\Pi = V(S_1, S_2, t) - \Delta_1 S_1 - \Delta_2 S_2$$

- application of multidimensional Ito calculus
- delta hedging with $\Delta_1 = \frac{\partial V}{\partial S_1}$ and $\Delta_2 = \frac{\partial V}{\partial S_2}$
- generalization of BS equation into 2D

$$-\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_1} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} = rV$$

$V = V(S_1, S_2, t; T, K, r, \sigma_1, \sigma_2, \rho)$ - value of option
$t,\ 0 \le t \le T$ - time to expiration, $T - t$ - current time

- reversal time $\Rightarrow$ initial condition is given by payoff function

$$\begin{aligned} V(S_1, S_2, 0) &= \max(\alpha_1 S_1 + \alpha_2 S_2 - K, 0) \quad \text{(call)} \\ V(S_1, S_2, 0) &= \max(K - \alpha_1 S_1 - \alpha_2 S_2, 0) \quad \text{(put)} \end{aligned}, \alpha_k > 0, \alpha_1 + \alpha_2 = 1$$

Basket options – numerical experiments

Parameter settings:

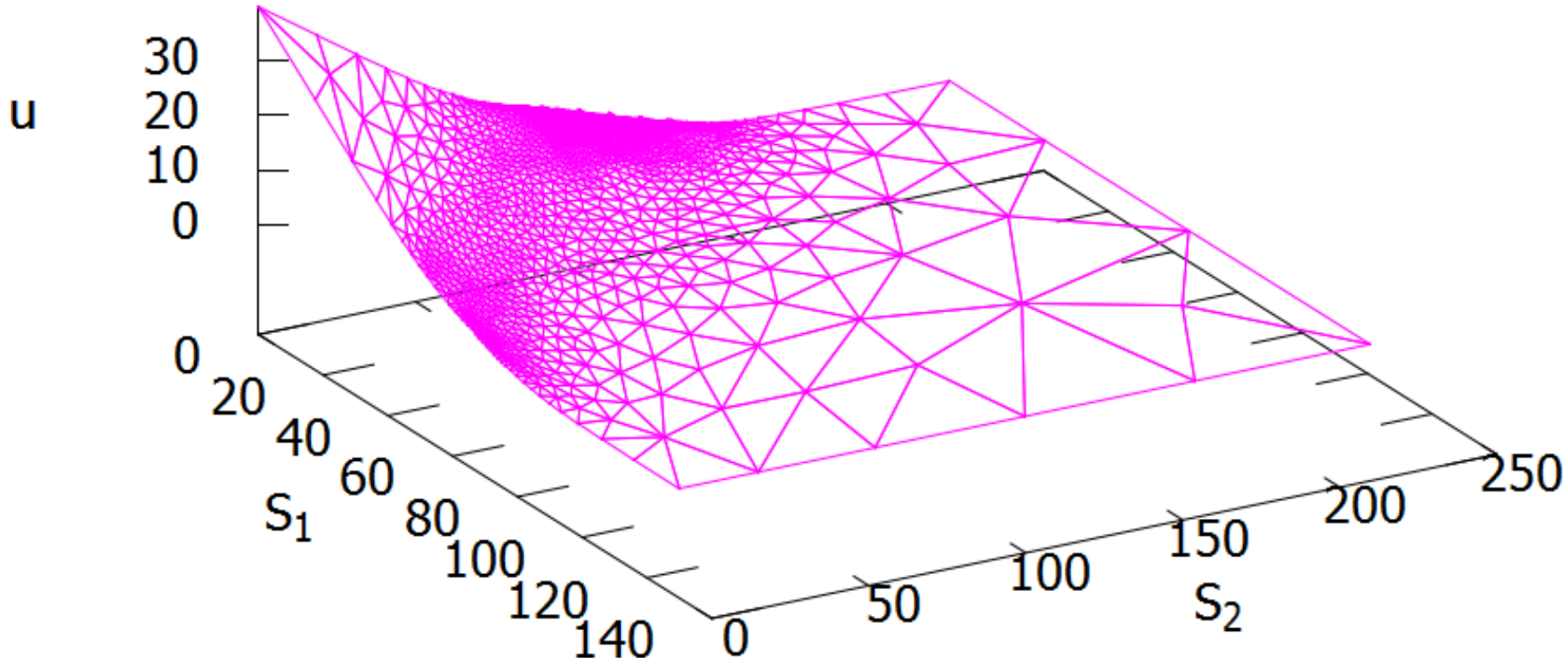
- basket put option with 60% Allianz ($\alpha_1 = 0.6$) and 40% Deutsche Bank ($\alpha_2 = 0.4$), strike at 40 Euro
- current date: September 13, 2011 (Tuesday),
- piecewise constatnt volatilities σ_1 and σ_2,
- expiration date $T = 0.257534$ (i.e. 94 days),
- risk-free interest rate $r = 0.01557$ p.a.,
- Pearson linear correlation $\rho = 0.88$
- maximal stock prices $S_1^{max} = 130.0$ and $S_2^{max} = 220.0$,
- two stocks trades at $S_1^{ref} = 59.79$ and $S_2^{ref} = 23.40$

Implementation settings:

- adaptively refined grid, constant time step $\tau = 1/365$
- piecewise P_1 (linear) approximations, sparse solver GMRES

Basket options - numerical results

- space-time plot of solution at maturity



Asian options

- be introduced to avoid the manipulation of prices on expiration date $\longrightarrow$ less sensitive to market fluctuations,
- a representative of the class of path dependent options,
- **value of Asian option** $V = V(S, A, t)$ depends also on the average A of the price of the underlying assets
- we distinguish

fixed strike	$\times$	continuous	$\times$	arithmetic averaging
floating strike		discrete		geometric averaging

- average of the stock price at time t is given by

$$A = \frac{1}{t} \int_0^t S(u)du$$

Asian options

- movement of the stock price is driven by SDE

$$dS(t) = rS(t)dt + \sigma S(t)dW(t)$$

- movement of the average-variable is driven by ODE

$$dA(t) = \frac{1}{t}\Big(S(t) - A(t)\Big)dt$$

- $r,\ r > 0$ - risk-free interest rate (constant),
- $\sigma,\ \sigma > 0$ - volatility of the price S (constant)
- analogous approach to 1D BS model with risk-free portfolio

$$\Pi = V(S, A, t) - \Delta S$$

- application of multidimensional Ito calculus
- delta hedging with $\Delta = \frac{\partial V}{\partial S}$
- we obtain 2D PDE model of Asian options

2D PDE model of Asian options

- continuous arithmetic Asian options with fixed strike

$$\frac{\partial V}{\partial t} - \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} - \frac{S-A}{T-t}\frac{\partial V}{\partial A} + rV = 0$$

$V = V(S, A, t; T, K, r, \sigma)$ - value of option
$t,\ 0 \le t \le T$ - time to expiration, $T - t$ - current time

- parabolic PDE degenerated in variable A
- reversal time $\Rightarrow$ initial condition is given by payoff function,

$$V(S, A, 0) = \max(A - K, 0) \quad \text{(call with fixed strike)}$$
$$V(S, A, 0) = \max(K - A, 0) \quad \text{(put with fixed strike)}$$

Asian options – numerical experiments

Parameter settings:

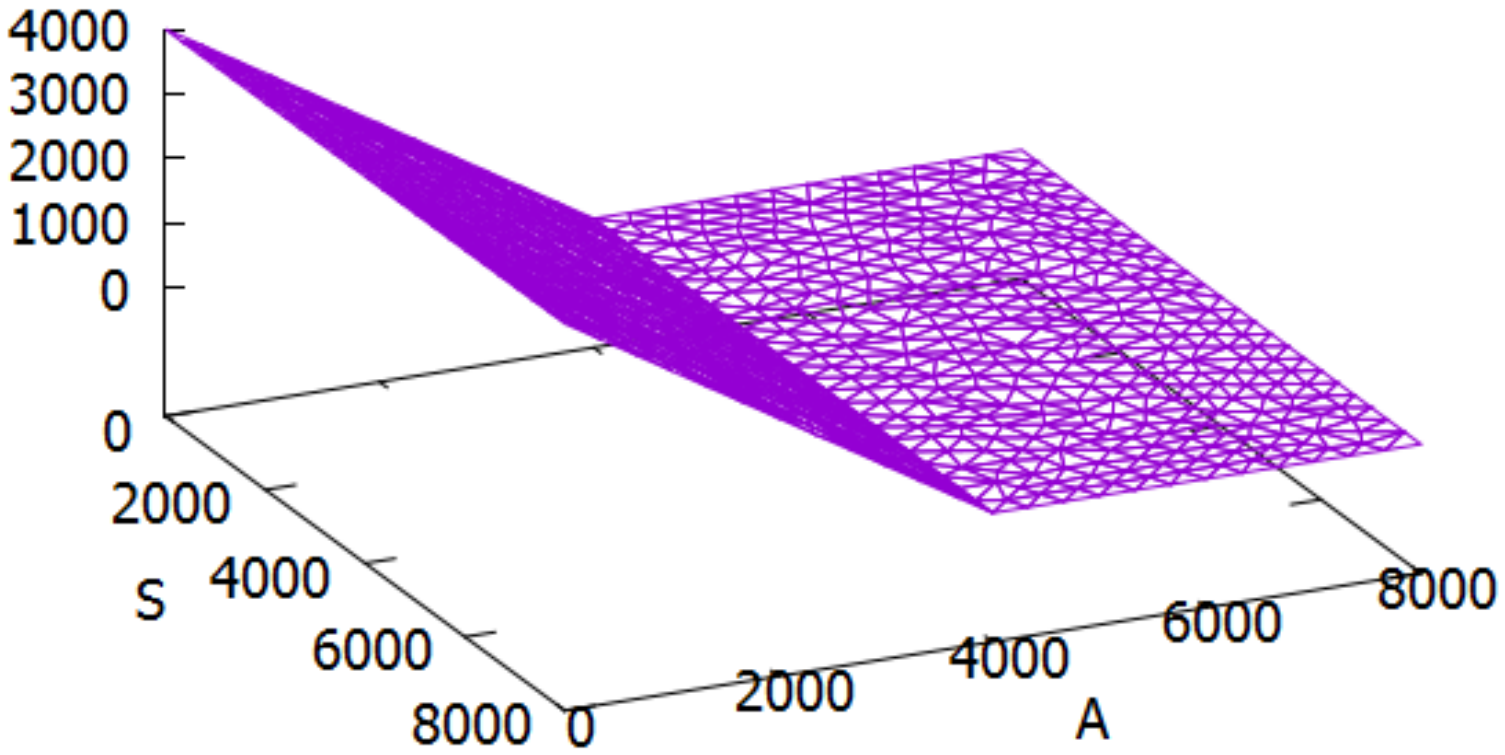
- Asian puts on German stock market index DAX,
- real data of the DAX on 15SEPT2011 with implied volatilities,
- expiration date $T = 0.50137$ (half year, 183 days),
- strike $K = 4000$,
- constant volatility $\sigma = 0.4594$,
- constant risk-free interest rate $r = 0.0176$,
- maximal stock price $S_{max} = A_{max} = 2K$ (usually),
- reference node $S_{ref} = 5508$

Implementation settings:

- uniform grid, constant time step $\tau = 1/365$,
- piecewise P_1 (linear) approximations, sparse solver GMRES

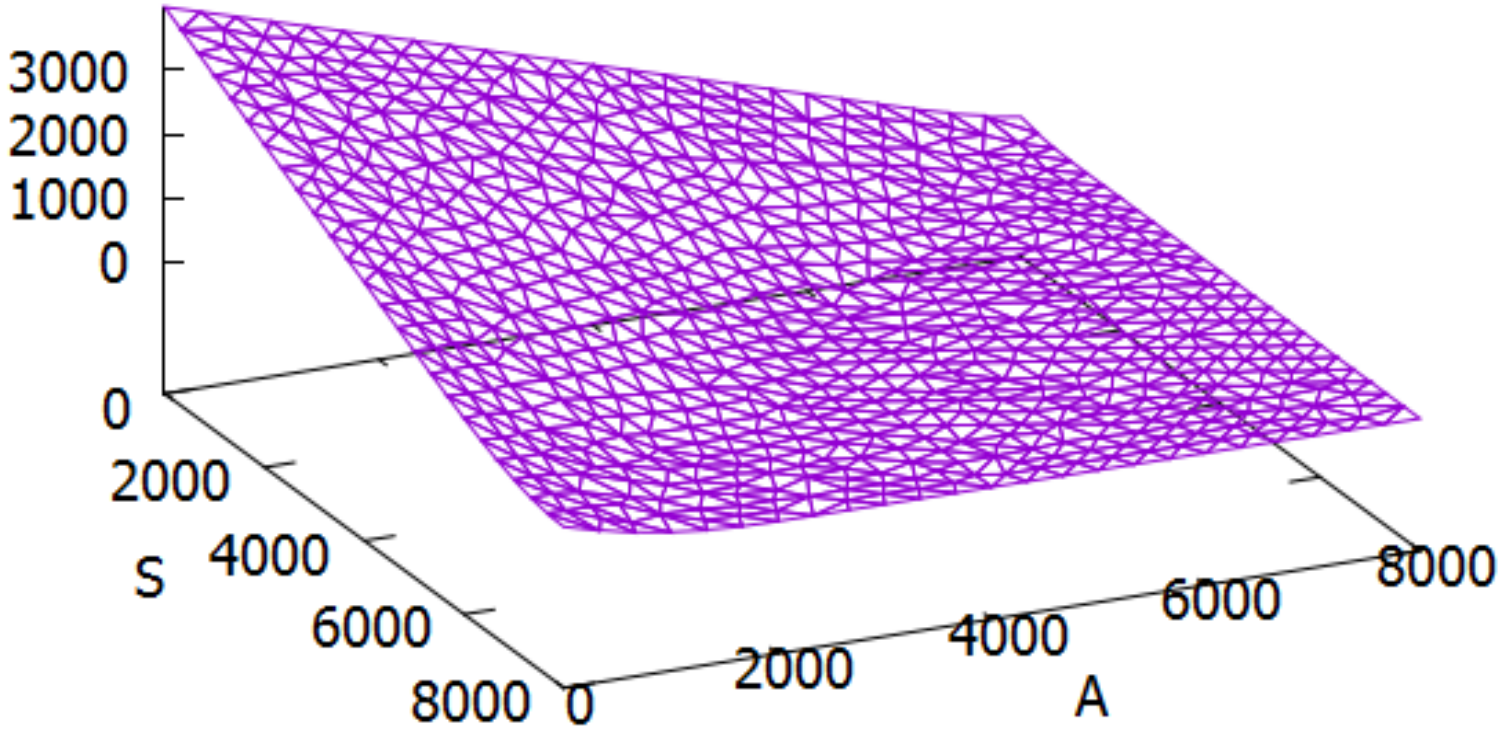
Asian options - numerical results (1)

- space-time plot of solution at $t = 0$ days (initial state)



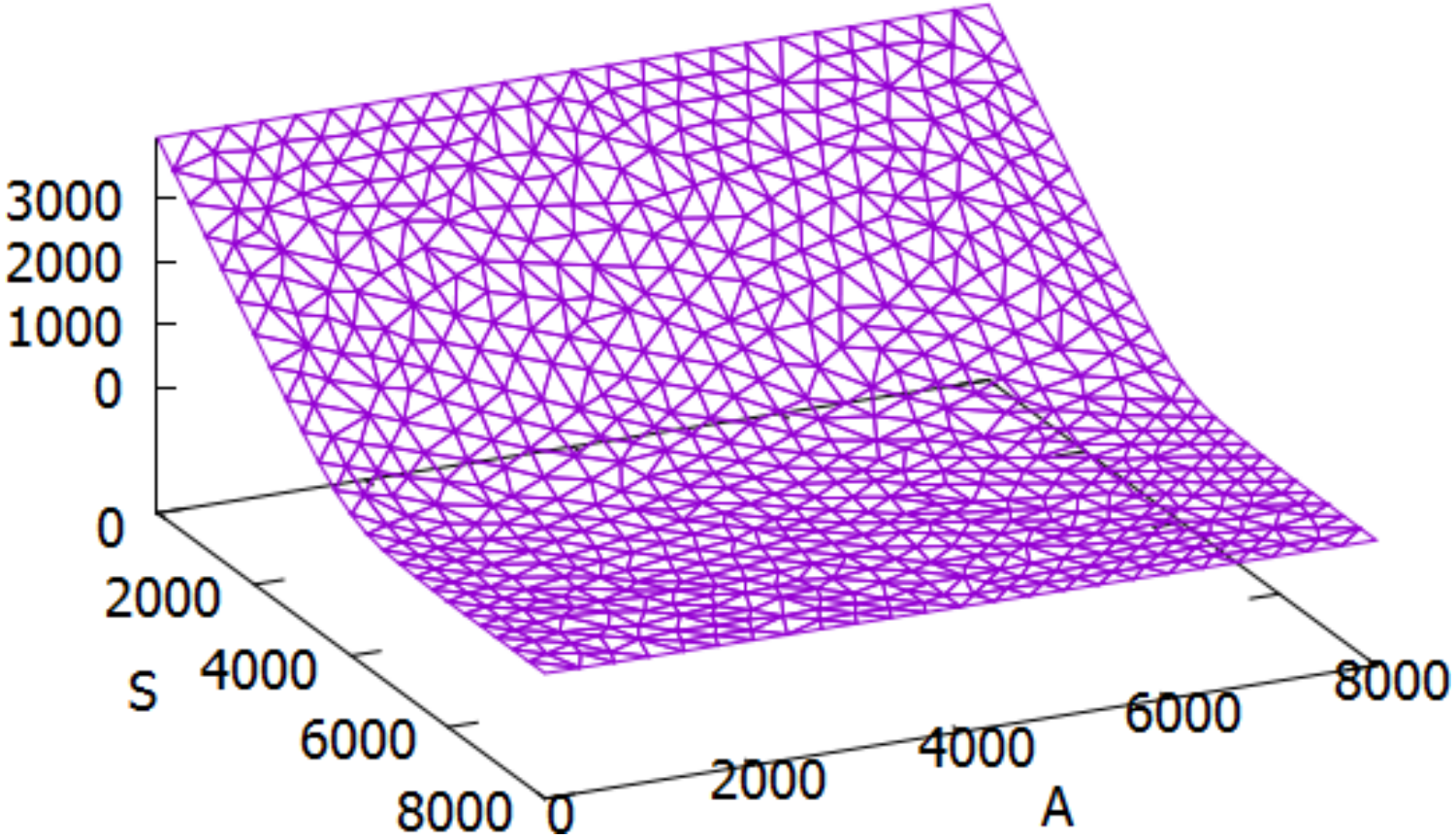
Asian options - numerical results (1)

- space-time plot of solution at $t = 92$ days (mid state)



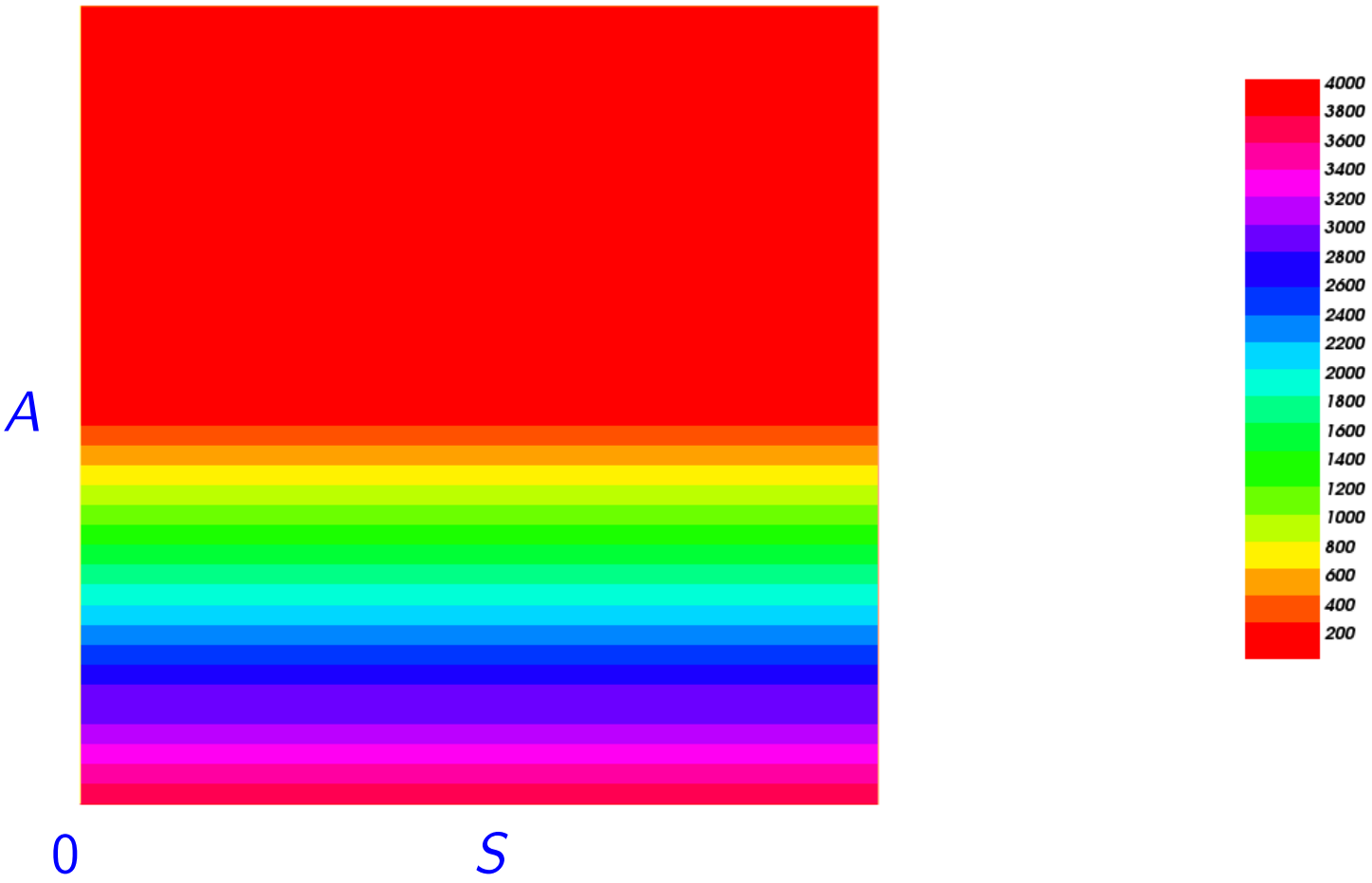
Asian options - numerical results (1)

- space-time plot of solution at $t = 183$ days (final state)



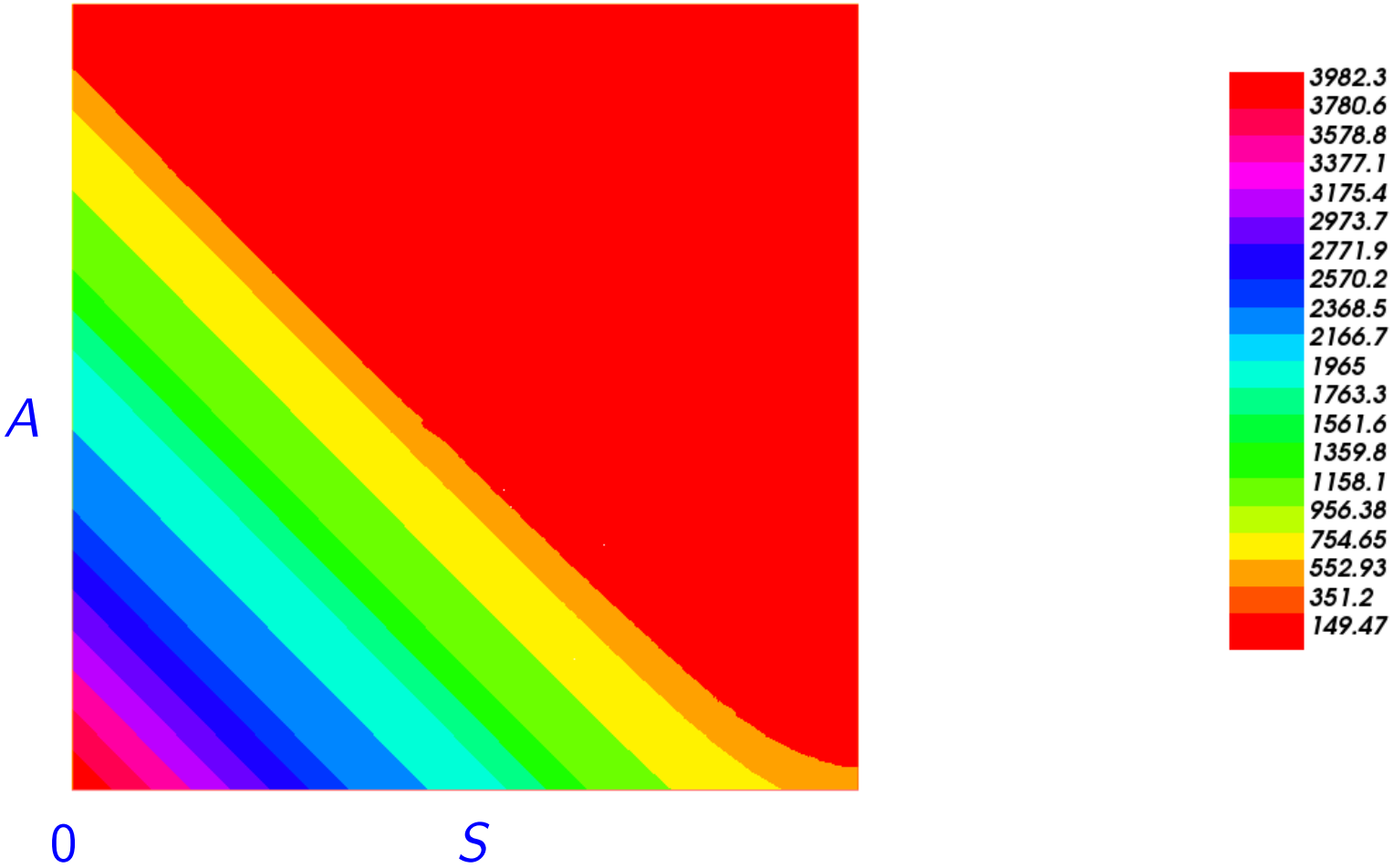
Asian options - numerical results (2)

- isovalues of solution at $t = 0$ days (initial state)



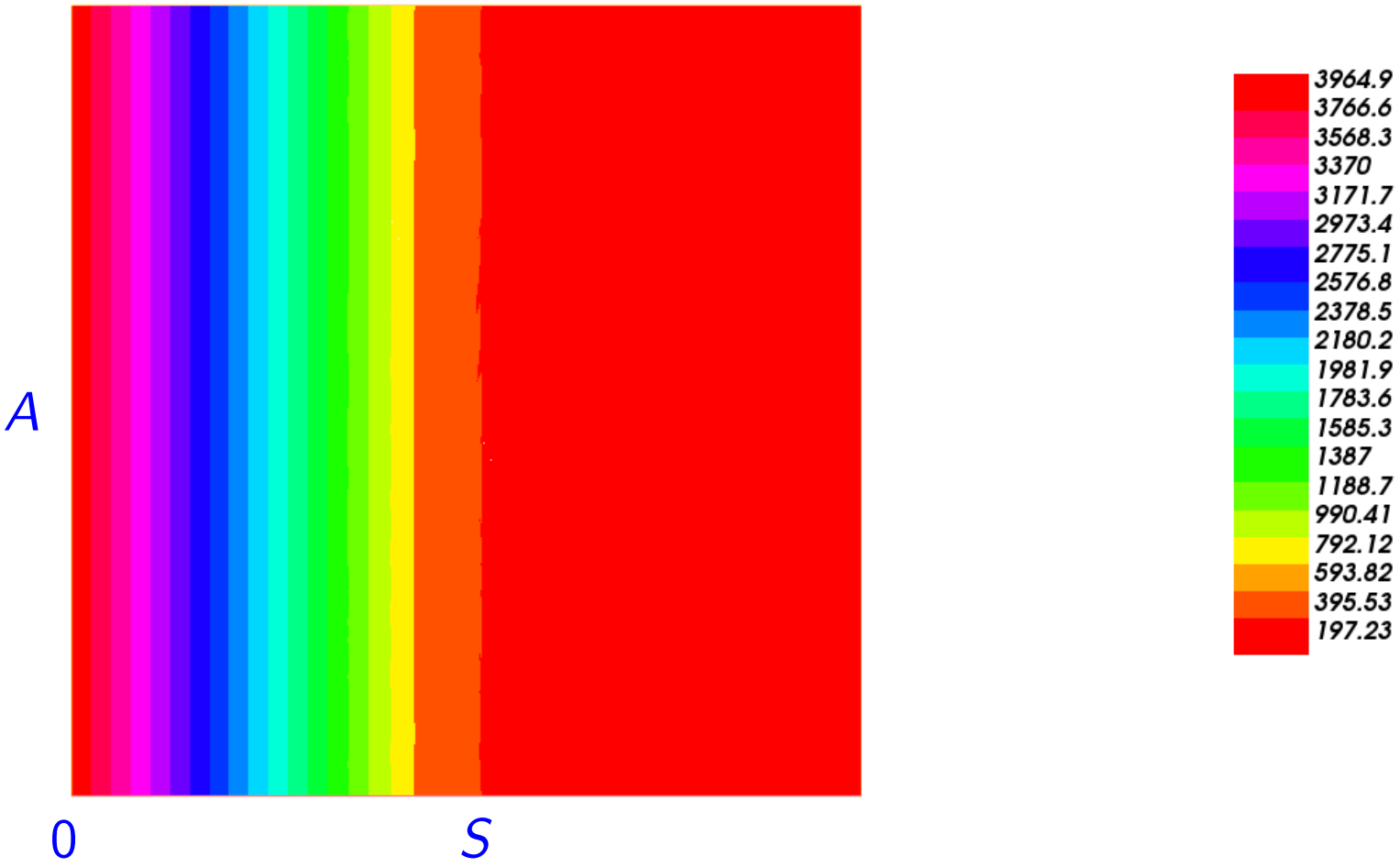
Asian options - numerical results (2)

- isovalues of solution at $t = 92$ days (mid state)



Asian options - numerical results (2)

- isovalues of solution at $t = 183$ days (final state)



1 Motivation

2 Path-dependent multi-asset options

3 Asian two-asset option with floating strike

4 Asian two-asset option with fixed strike

General model of path-dependent multi-asset options

- combination of basket and path-dependent options
- simplification: basket options with 2 assets
- price function $V = V(S_1(t), S_2(t), A(t), t)$ depends on two correlated price processes

$$dS_k(t) = rS_k(t)dt + \sigma_k S_k(t)dW_k(t), \quad k = 1, 2$$

and on path-dependent variable (in general form)

$$A(t) = \frac{1}{t}\int_0^t g(S_1(u), S_2(u), u)du \quad \text{(continuous case)}$$

driven by ODE (with drift component only)

$$dA(t) = \frac{1}{t}\Big(g(S_1(t), S_2(t), t) - A(t)\Big)dt$$

where g is given according to specific type of option

General model of path-dependent multi-asset options

- analogous approach to basket and Asian options with portfolio
- application of multidimensional Ito calculus
- delta hedging with $\Delta_1 = \frac{\partial V}{\partial S_1}$ and $\Delta_2 = \frac{\partial V}{\partial S_2}$
- 3D PDE model of path-dependent two-asset options

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_1} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}$$
$$+ rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} + \frac{1}{t}\Big(g(S_1, S_2, t) - A\Big)\frac{\partial V}{\partial A} - rV = 0$$

$V = V(S_1, S_2, A, t; T, K, r, \sigma_1, \sigma_2, \rho)$ - value of option
t, $0 \le t \le T$ - current time, $T - t$ - time to expiration

- parabolic PDE degenerated in variable A
- backward PDE $\Rightarrow$ terminal condition is given by payoff

Asian two-asset basket option

- various forms of $g \longrightarrow$ different path-dependent options
- Asian option = average options (subclass of path-dep. opt.)
- continuous arithmetic averaging

$$A(t) = \frac{1}{t}\int_0^t \Big(\alpha_1 S_1(u) + \alpha_2 S_2(u)\Big)\,du$$

- continuous geometric averaging (not considered here)

$$A(t) = \frac{1}{t}\exp\left(\int_0^t \Big(\alpha_1 S_1(u) + \alpha_2 S_2(u)\Big)\,du\right)$$

- continuous weighted arithmetic averaging

$$A(t) = \frac{1}{t}\int_0^t a(t-u)\Big(\alpha_1 S_1(u) + \alpha_2 S_2(u)\Big)\,du$$

where $\alpha_1, \alpha_2 > 0$ are weights with $\alpha_1 + \alpha_2 = 1$
and $a(\cdot) \geq 0$ s.t. $\int_0^\infty a(\xi)d\xi < \infty$

Asian two-asset basket option: payoff function

- function of average over some time period prior to expiry
- 2 ways how the average is incorporated into payoff
  - average strike option (= option with floating strike)

$$V(S, A, T) = \max(\alpha_1 S_1(T) + \alpha_2 S_2(T) - A(T), 0) \quad \text{(call)}$$
$$V(S, A, T) = \max(A(T) - \alpha_1 S_1(T) - \alpha_2 S_2(T), 0) \quad \text{(put)}$$

  - average rate option (= option with fixed strike)

$$V(S, A, T) = \max(A(T) - K, 0) \quad \text{(call)}$$
$$V(S, A, T) = \max(K - A(T), 0) \quad \text{(put)}$$

- we focus on

Asian two-asset basket option with floating strike Asian two-asset basket option with fixed strike

1 Motivation

2 Path-dependent multi-asset options

3 Asian two-asset option with floating strike

4 Asian two-asset option with fixed strike

3D PDE model

- continuous arithmetic Asian two-asset basket option with floating strike

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_1} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}$$
$$+ rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} + \frac{1}{t}\Big(\alpha_1 S_1 + \alpha_2 S_2 - A\Big)\frac{\partial V}{\partial A} - rV = 0$$

$V = V(S_1, S_2, A, t; T, K, r, \sigma_1, \sigma_2, \rho)$ - value of option
t, $0 \le t \le T$ - current time, $T - t$ - time to expiration
$S_1, S_2, A > 0$ - underlying assets and average

- backward PDE $\Rightarrow$ terminal condition is given by payoff,
- for $t = 0$ we obtain 2D Black-Scholes model

3D PDE model - boundary conditions (1)

- localization problem on bounded domain
- at far-field boundary (from asymptotic behaviour at infinity)

$$\text{if } V \text{ is call} \begin{cases} \lim_{S_1\to\infty} \frac{\partial V}{\partial S_1}(S_1, S_2, A, t) = \alpha_1, \\ \lim_{S_2\to\infty} \frac{\partial V}{\partial S_2}(S_1, S_2, A, t) = \alpha_2, \\ \lim_{A\to\infty} V(S_1, S_2, A, t) = 0, \end{cases}$$

and

$$\text{if } V \text{ is put} \begin{cases} \lim_{S_1\to\infty} V(S_1, S_2, A, t) = 0, \\ \lim_{S_2\to\infty} V(S_1, S_2, A, t) = 0, \\ \lim_{A\to\infty} \frac{\partial V}{\partial A}(S_1, S_2, A, t) = 1, \end{cases}$$

3D PDE model - boundary conditions (2)

- on planes $S_1 = 0$ and $S_2 = 0$ (concept of extrapolated b.c.)

$$\text{if } V \text{ is call} \begin{cases} V(0, S_2, A, t) := \lim_{\varepsilon \to 0+} V(\varepsilon, S_2, A, t), \\ V(S_1, 0, A, t) := \lim_{\varepsilon \to 0+} V(S_1, \varepsilon, A, t) \end{cases}$$

and

$$\text{if } V \text{ is put} \begin{cases} \frac{\partial V}{\partial S_1}(0, S_2, A, t) := \lim_{\varepsilon \to 0+} \frac{\partial V}{\partial S_1}(\varepsilon, S_2, A, t), \\ \frac{\partial V}{\partial S_2}(S_1, 0, A, t) := \lim_{\varepsilon \to 0+} \frac{\partial V}{\partial S_2}(S_1, \varepsilon, A, t) \end{cases}$$

- only one boundary condition in A-direction is prescribed

Dimensional reduction

- to avoid the higher complexity of 3D model
- change of variables (approach from [Ingersoll])

$$x_1 = \frac{S_1}{A}, \qquad x_2 = \frac{S_2}{A}, \qquad \hat{t} = T - t$$

- transformation of price function

$$u(x, \hat{t}) = \frac{1}{A} V(S_1, S_2, A, t), \quad \text{where } x = [x_1, x_2]$$

- transformation of payoffs (= initial conditions)

$$u(x, 0) = \tfrac{1}{A} V(S_1, S_2, A, T) = \max(\alpha_1 x_1 + \alpha_2 x_2 - 1, 0) \quad \text{(call)}$$
$$u(x, 0) = \tfrac{1}{A} V(S_1, S_2, A, T) = \max(1 - \alpha_1 x_1 - \alpha_2 x_2, 0) \quad \text{(put)}$$

- transformation of PDE: homogeneity w.r.t. A
  $\longrightarrow$ separation of variable A $\longrightarrow$ dimensional reduction from 3D to 2D

2D reformulation

- localization on a truncated computational domain:

$$[x_1, x_2] \in \boldsymbol{R}_0^2 \longrightarrow [x_1, x_2] \in \Omega = (0, x_1^{max}) \times (0, x_2^{max})$$

- convection-diffusion-reaction equation in divergence-free form

We seek $u : Q_T = \Omega \times (0, T) \to \boldsymbol{R}$ such that

$$\frac{\partial u}{\partial \hat{t}} - \operatorname{div}(\boldsymbol{D}(x) \cdot \nabla u) + \nabla \cdot \vec{f}(x, \hat{t}, u) + \gamma(x, \hat{t}) u = 0 \quad \text{in } Q_T, \quad (2)$$

$$B(u) = 0 \quad \text{on } \partial\Omega,$$

$$u(x, 0) = u^0(x), \quad x \in \Omega,$$

where

$$\boldsymbol{D}(x) \equiv \{D(x)_{kl}\}_{k,l=1}^2 = \frac{1}{2} \begin{pmatrix} \sigma_1^2 x_1^2 & \rho\sigma_1\sigma_2 x_1 x_2 \\ \rho\sigma_1\sigma_2 x_1 x_2 & \sigma_2^2 x_2^2 \end{pmatrix},$$

2D reformulation

- localization on a truncated computational domain:

$$[x_1, x_2] \in \boldsymbol{R}_0^2 \longrightarrow [x_1, x_2] \in \Omega = (0, x_1^{max}) \times (0, x_2^{max})$$

- convection-diffusion-reaction equation in divergence-free form

We seek $u : Q_T = \Omega \times (0, T) \to \boldsymbol{R}$ such that

$$\frac{\partial u}{\partial \hat{t}} - \operatorname{div}(\boldsymbol{D}(x) \cdot \nabla u) + \nabla \cdot \vec{f}(x, \hat{t}, u) + \gamma(x, \hat{t}) u = 0 \quad \text{in } Q_T, \quad (2)$$

$$B(u) = 0 \quad \text{on } \partial\Omega,$$

$$u(x, 0) = u^0(x), \quad x \in \Omega,$$

where $\vec{f}(x, \hat{t}, u) = (f_1(x, \hat{t}, u), f_2(x, \hat{t}, u))^T$ with

$$f_k(x, \hat{t}, u) = \left(\sigma_k^2 + \frac{1}{2}\rho\sigma_1\sigma_2 - r + \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T - \hat{t}} \right) x_k \cdot u, \; k = 1, 2$$

2D reformulation

- localization on a truncated computational domain:

$$[x_1, x_2] \in \boldsymbol{R}_0^2 \longrightarrow [x_1, x_2] \in \Omega = (0, x_1^{max}) \times (0, x_2^{max})$$

- convection-diffusion-reaction equation in divergence-free form

We seek $u : Q_T = \Omega \times (0, T) \to \boldsymbol{R}$ such that

$$\frac{\partial u}{\partial \hat{t}} - \operatorname{div}(\boldsymbol{D}(x) \cdot \nabla u) + \nabla \cdot \vec{f}(x, \hat{t}, u) + \gamma(x, \hat{t}) u = 0 \quad \text{in } Q_T, \quad (2)$$

$$B(u) = 0 \quad \text{on } \partial\Omega,$$

$$u(x, 0) = u^0(x), \quad x \in \Omega,$$

where

$$\gamma(x, \hat{t}) = 3r - \frac{4\alpha_1 x_1 + 4\alpha_2 x_2 - 3}{T - \hat{t}} - \sigma_1^2 - \sigma_2^2 - \rho\sigma_1\sigma_2$$

2D reformulation

- localization on a truncated computational domain:

$$[x_1, x_2] \in \boldsymbol{R}_0^2 \longrightarrow [x_1, x_2] \in \Omega = (0, x_1^{max}) \times (0, x_2^{max})$$

- convection-diffusion-reaction equation in divergence-free form

We seek $u : Q_T = \Omega \times (0, T) \to \boldsymbol{R}$ such that

$$\frac{\partial u}{\partial \hat{t}} - \operatorname{div}(\boldsymbol{D}(x) \cdot \nabla u) + \nabla \cdot \vec{f}(x, \hat{t}, u) + \gamma(x, \hat{t}) u = 0 \quad \text{in } Q_T, \quad (2)$$

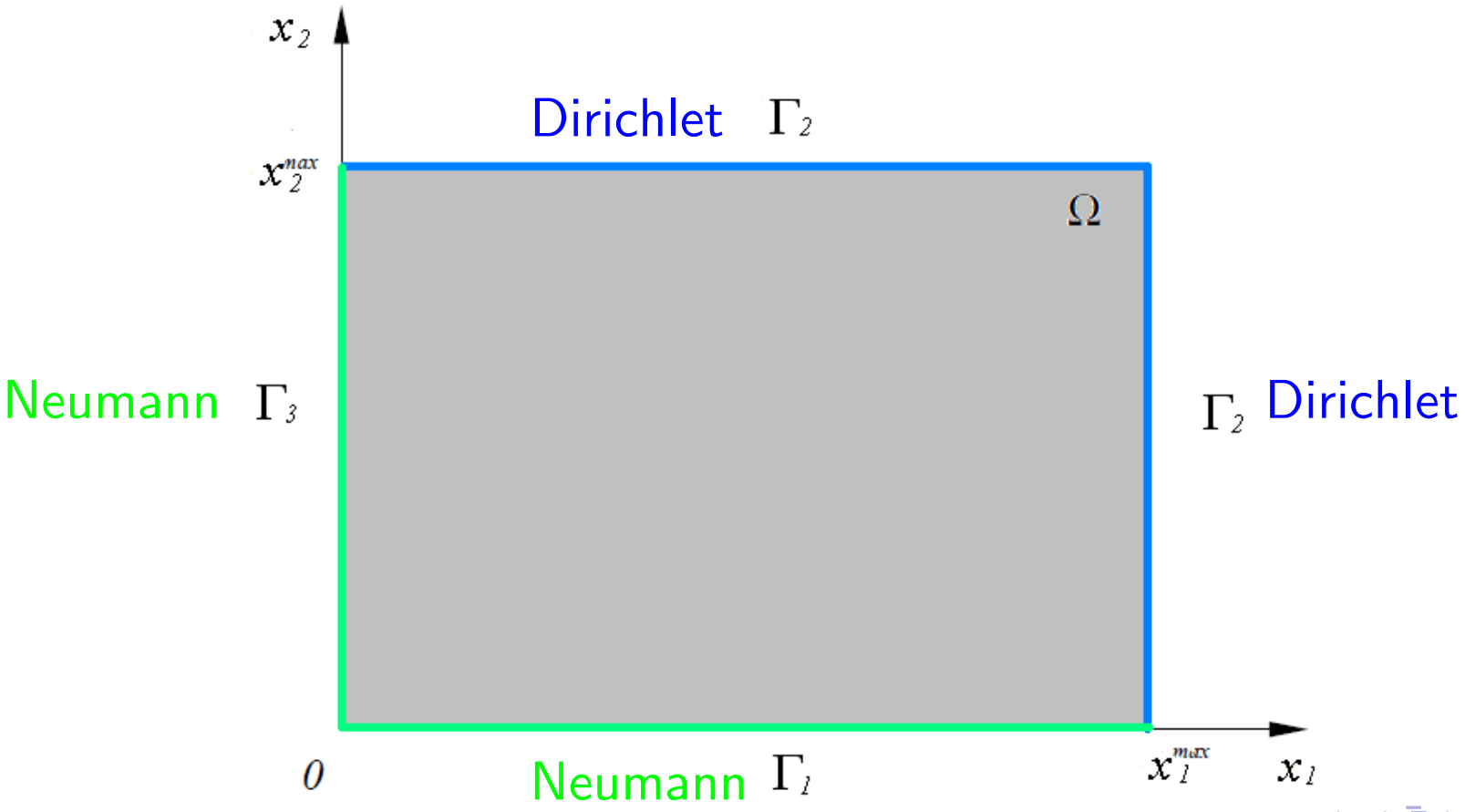
$$B(u) = 0 \quad \text{on } \partial\Omega,$$

$$u(x, 0) = u^0(x), \quad x \in \Omega,$$

- $\hat{t}$ - time to maturity $\longrightarrow$ i.c. given by payoffs u^0
- boundary conditions $B(u)$ specified latter

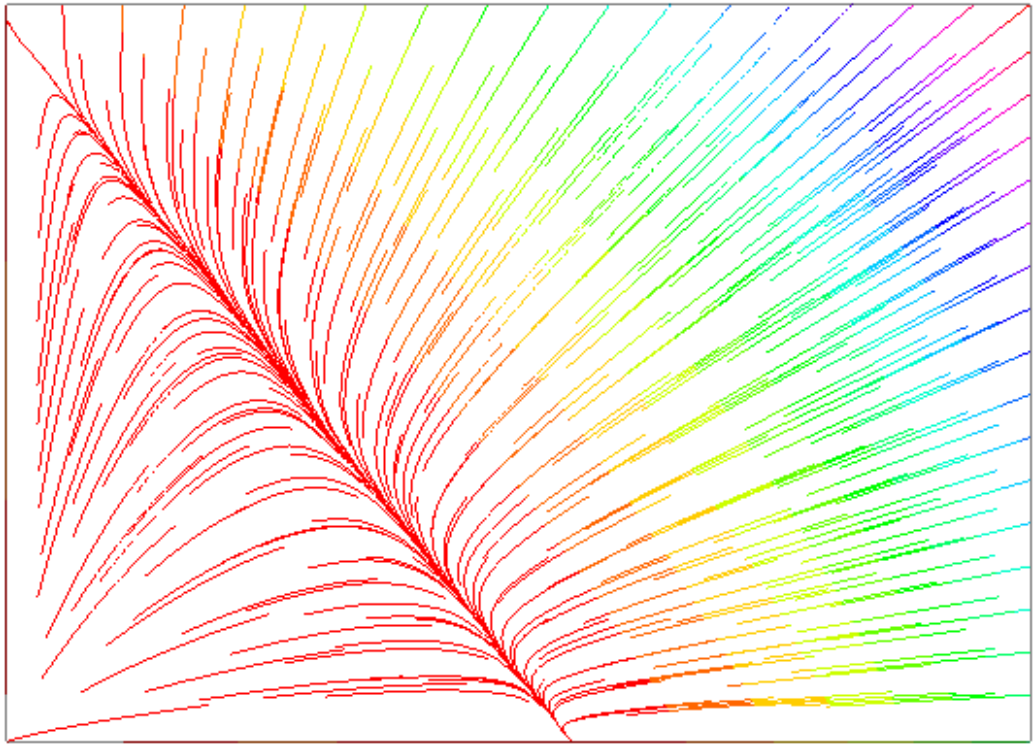
Boundary conditions (1)

Let $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. W.l.o.g. we consider the case of the put option.



Boundary conditions (2)

- setting boundary conditions in consistency with the vector field induced by physical fluxes, i.e. $\left(\frac{\partial f_1}{\partial u}, \frac{\partial f_2}{\partial u}\right)$



Boundary conditions (3)

- reformulation of original b.c.
  - homogeneous Neumann b.c. on axes $x_1 = 0$ and $x_2 = 0$

$$\left(\boldsymbol{D}(x) \cdot \nabla u(x_1, x_2, \hat{t})\right) \cdot \vec{n} = 0, \quad \text{on } \Gamma_i, i = 1, 3, \ \hat{t} > 0,$$

  where $\vec{n}$ is the unit outer normal vector,
  - asymptotic behaviour of options at infinity

$$u(x_1, x_2, \hat{t}) = 0 \quad \text{on } \Gamma_2, \ \hat{t} \in (0, T).$$

- b.c. for calls follow from put-call parity

Weighted Lebesgue space

- standard Lebesgue space

$$L^2(\Omega) = \left\{ v \in L^1_{loc}(\Omega) : \int_\Omega v^2(x)\mathrm{d}x < +\infty \right\}$$

with scalar product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$:

$$(u,v) = \int_\Omega u(x)\cdot v(x)\mathrm{d}x, \qquad \|u\| = \sqrt{(u,u)}$$

- w is weight, i.e. $w : \boldsymbol{R}^2 \to [0,\infty)$, $w \in L^1_{loc}(\Omega)$ and $w \neq 0$,
- weighted Lebesgue space

$$L^2_w = L^2(\Omega, w) = \left\{ v \in L^1_{loc}(\Omega) : \int_\Omega \sqrt{w}\cdot v \in L^2(\Omega) \right\},$$

with scalar product $(\cdot,\cdot)_w$ and induced norm $\|\cdot\|_w$:

$$(u,v)_w = \int_\Omega u(x)\cdot v(x)\cdot w(x)\mathrm{d}x, \qquad \|u\|_w = \|\sqrt{w}\cdot u\|.$$

Weighted Sobolev space

- weighted Sobolev space

$$H^1_w = H^1(\Omega, w_1, w_2) = \left\{ v \in L^2(\Omega) : \sqrt{w_i} \cdot \frac{\partial v}{\partial x_i} \in L^2(\Omega),\ i = 1, 2 \right\}$$

with the norm

$$\|u\|_{1,w} = \left(\|u\|^2 + \left\| \frac{\partial v}{\partial x_1} \right\|^2_{w_1} + \left\| \frac{\partial v}{\partial x_2} \right\|^2_{w_2} \right)^{1/2}$$

- in our case, we take: $w_1(x) = x_1^2$ and $w_2(x) = x_2^2$,
- to treat with b.c., we introduce space:

$$H^1_{w,0} = \left\{ v \in H^1_w : v\big|_{\Gamma_D} = 0 \right\}$$

Properties of H^1_w

The space H^1_w has the following properties:

(a) H^1_w is separable,

(b) $\mathcal{D}(\overline{\Omega}) = \{v \in C_0^\infty(\boldsymbol{R}^2) : v|_\Omega\}$ is dense in H^1_w,

(c) H^1_w is dense in $L^2(\Omega)$ with continuous embeding $H^1_w \subseteq L^2(\Omega)$,

(d) the seminorm

$$|u|_{1,w} = \left(\sum_{i=1}^{2} \left\| \frac{\partial v}{\partial x_i} \right\|_{w_i}^2 \right)^{1/2}$$

is in fact a norm on $H^1_{w,0}$ equivalent to $\|\cdot\|_{1,w}$.

Variational formulation (1)

- we define linear partial differential operator

$$\mathcal{L}_t u = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\boldsymbol{D}(x)\cdot\nabla u\right) + \sum_{i=1}^{2} a_i(x,\hat{t})\frac{\partial u}{\partial x_i} + \left(r - \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T-\hat{t}}\right) u.$$

where

$$a_1(x,\hat{t}) = \left(\sigma_1^2 + \frac{1}{2}\rho\sigma_1\sigma_2 - r + \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T-\hat{t}}\right) x_1,$$

$$a_2(x,\hat{t}) = \left(\sigma_2^2 + \frac{1}{2}\rho\sigma_1\sigma_2 - r + \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T-\hat{t}}\right) x_2.$$

Variational formulation (2)

- standard approach leads to

$$\left(\frac{\partial u}{\partial \hat{t}}, v\right) + (\mathcal{L}_t u, v) = 0 \quad \forall\, v \in H^1_{w,0},\ \text{a.e. } \hat{t} \in (0, T),$$

where

$$\begin{aligned}(\mathcal{L}_t u, v) \quad = \quad & \int_\Omega \boldsymbol{D}(x) \cdot \nabla u \cdot \nabla v \,\mathrm{d}x + \sum_{i=1}^{2} \int_\Omega a_i(x, \hat{t}) \frac{\partial u}{\partial x_i}\, v \,\mathrm{d}x \\ & + \int_\Omega \left(r - \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T - \hat{t}} \right) uv \,\mathrm{d}x.\end{aligned}$$

Weak solution

- assume that there exists a function

$$u_g \in C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_w),\ \frac{\partial u_g}{\partial \hat{t}} \in L^2(0,T;H'_w)$$

with $u_g\big|_{\Gamma_D} = g$ (prescribed Dirichlet b.c.)

Find u: $u - u_g \in L^2(0,T;H^1_{w,0})$, $u \in C^0([0,T];L^2(\Omega))$, s.t. $\frac{\partial u}{\partial \hat{t}} \in L^2(0,T;H'_w)$ satisfying $u\big|_{\hat{t}=0} = u^0$ in Ω ($u^0 \in L^2(\Omega)$) and

$$\left(\frac{\partial u}{\partial \hat{t}}, v\right) + (\mathcal{L}_t u, v) = 0 \quad \forall\, v \in H^1_{w,0},\ \text{a.e. } \hat{t} \in (0,T), \qquad (3)$$

Properties of $\mathcal{L}_t$

- boundedness: $\exists\, C_B = C_B(\hat{t}) > 0$ s.t.

$$(\mathcal{L}_t u, v) \leq C_B(\hat{t})|u|_{1,w}|v|_{1,w}, \quad \forall u, v \in H^1_{w,0},\ \hat{t} \in (0, T^*]$$

- Gårding inequality: $\exists\, C_G = C_G(\hat{t}), C_g = C_g(\hat{t}) > 0$ s.t.

$$(\mathcal{L}_t u, u) \geq C_G(\hat{t})|u|^2_{1,w} - c_g(\hat{t})\|u\|^2, \quad \forall u \in H^1_{w,0},\ \hat{t} \in (0, T^*]$$

- Gårding inequality $\Rightarrow$ ellipticity (strict positivity) via easy transformation $u = e^{\lambda \hat{t}} \cdot z$, $\lambda = \max_{\hat{t} \in [0, T^*]} c_g(\hat{t})$, $z \in H^1_w$, i.e.

$$\left(\frac{\partial z}{\partial \hat{t}}, v\right) + (\mathcal{L}_t z, v) + \lambda(z, v) = 0 \quad \forall\, v \in H^1_{w,0},\ \text{a.e. } \hat{t} \in (0, T^*),$$

with $z\big|_{\hat{t}=0} = u^0 \in L^2(\Omega)$ and $0 < T^* < T$.

Unique solvability

Theorem

Problem (3) has a unique weak solution.

Proof (sketch 1/2)

Abstract theory: Let H and V be Hilbert spaces such that V is embedded continuously and densely in H. Let $\mathcal{A}_t(\cdot,\cdot))$ be continuous bilinear form on $V \times V$. The abstract variational parabolic (AVP) problem associated to the triple $(H, V, \mathcal{A}_t(\cdot,\cdot))$ is the following: Given $f(\hat{t}) \in L^2(0,T,V')$ and $u^0 \in H$, find $u \in W(0,T,V,V') = \{u \in L^2(0,T,V) : \frac{du}{d\hat{t}} \in L^2(0,T,V')\}$ s.t.

$$\frac{\partial}{\partial \hat{t}}(u(\hat{t}), v)_H + \mathcal{A}_t(u(\hat{t}), v) = \langle f(\hat{t}), v\rangle \quad \forall v \in V,$$
$$u(0) = u^0.$$

Unique solvability

Theorem

Problem (3) has a unique weak solution.

Proof (sketch 2/2)

If the form $\mathcal{A}_t(\cdot,\cdot)$ is coercive, then AVP problem admits a unique solution for all $u^0 \in H$.

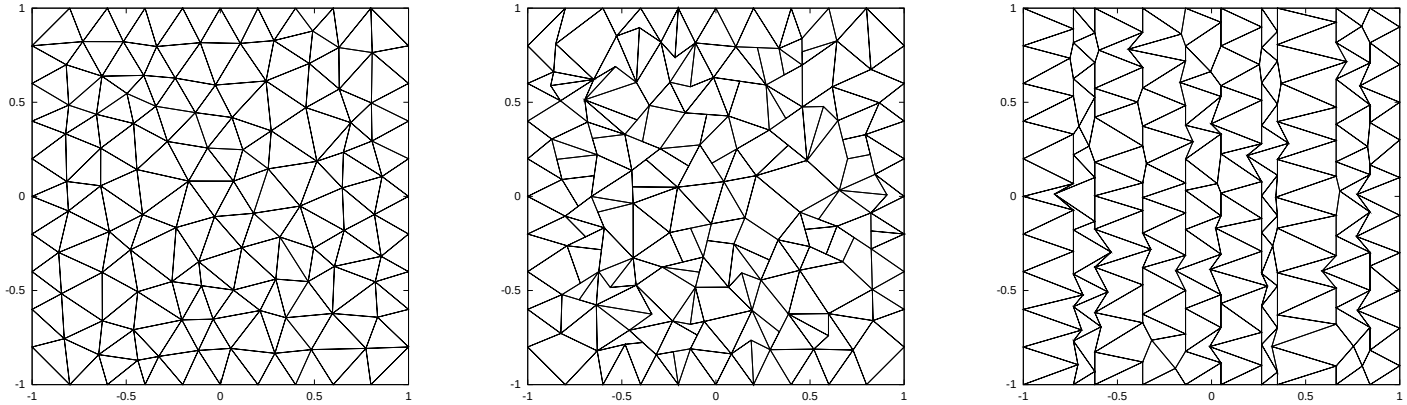
In our case, we take

$$
\begin{aligned}
&T = T^* \\
&H = L^2(\Omega), \quad V = H^1_{w,0} \\
&\langle f(\hat{t}), v\rangle = -\frac{\partial}{\partial \hat{t}}(u_g(\hat{t}), v)_H - \mathcal{A}_t(u_g(\hat{t}), v), \\
&\mathcal{A}_t(u, v) = (\mathcal{L}_t u, v)
\end{aligned}
$$

Discontinuous Galerkin approximations

- exact solution of (2) is difficult to find,
- we construct partitions $\mathcal{T}_h$ of Ω,
- we define space $S_{hp} \approx H^1_w(\Omega)$ over $\mathcal{T}_h$ with $dim(S_{hp}) < \infty$
- we derive DG semi-discrete formulation,
- we discretize in temporal variable,
- we obtain fully discrete DG numerical scheme,
- we construct basis of space S_{hp},
- we assemble a system matrix and right-hand side,
- we solve system of linear algebraic equations,
- we obtain DG solution at each time level.

Triangulations



- let $\mathcal{T}_h$, $h > 0$ be a partition of $\overline{\Omega}$,
- $\mathcal{T}_h = \{K\}_{K \in \mathcal{T}_h}$, K are polygons (nonconvex, hanging nodes),
- $h_K = \mathrm{diam}(K)$, $h = \max_{K \in \mathcal{T}_h} h_K$,
- let $\mathcal{F}_h = \{\Gamma\}_{\Gamma \in \mathcal{F}_h}$ be a set of all edges of $\mathcal{T}_h$,
- we distinguish

$$\left.\begin{array}{l} \text{a) inner edges } \mathcal{F}_h^I \\ \text{b) Dirichlet edges } \mathcal{F}_h^D \\ \text{c) Neumann edges } \mathcal{F}_h^N \end{array}\right\} \Rightarrow \mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^D \cup \mathcal{F}_h^N,$$

Spaces of discontinuous functions

- let $p_K \geq 1, K \in \mathcal{T}_h$ be local polynomial degree,
- we set vector $\mathbf{p} \equiv \{p_K, K \in \mathcal{T}_h\}$,
- over $\mathcal{T}_h$ we define:
  - *broken weighted Sobolev space*

$$H^1_w(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^1_w(K) \; \forall \, K \in \mathcal{T}_h\},$$

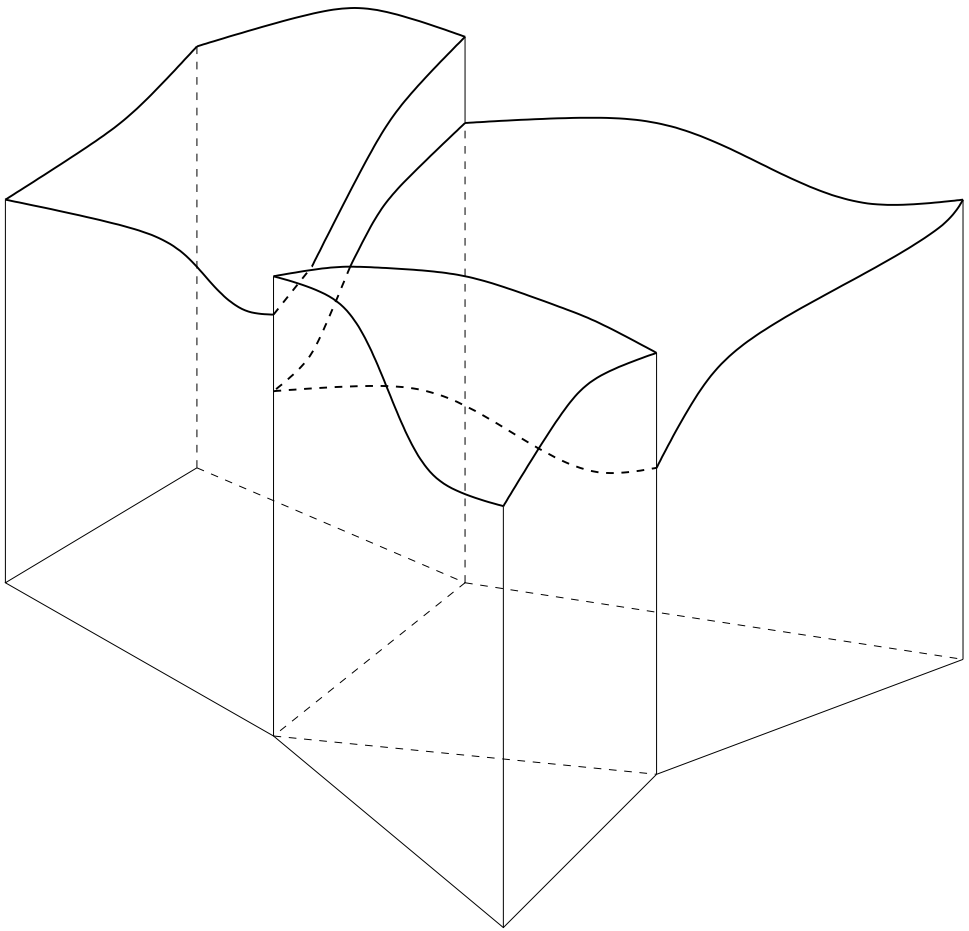
where

$$H^1_w(K) = \left\{ v \in L^2(K); x_1 \frac{\partial v}{\partial x_1} \in L^2(K) \wedge x_2 \frac{\partial v}{\partial x_2} \in L^2(K) \right\},$$

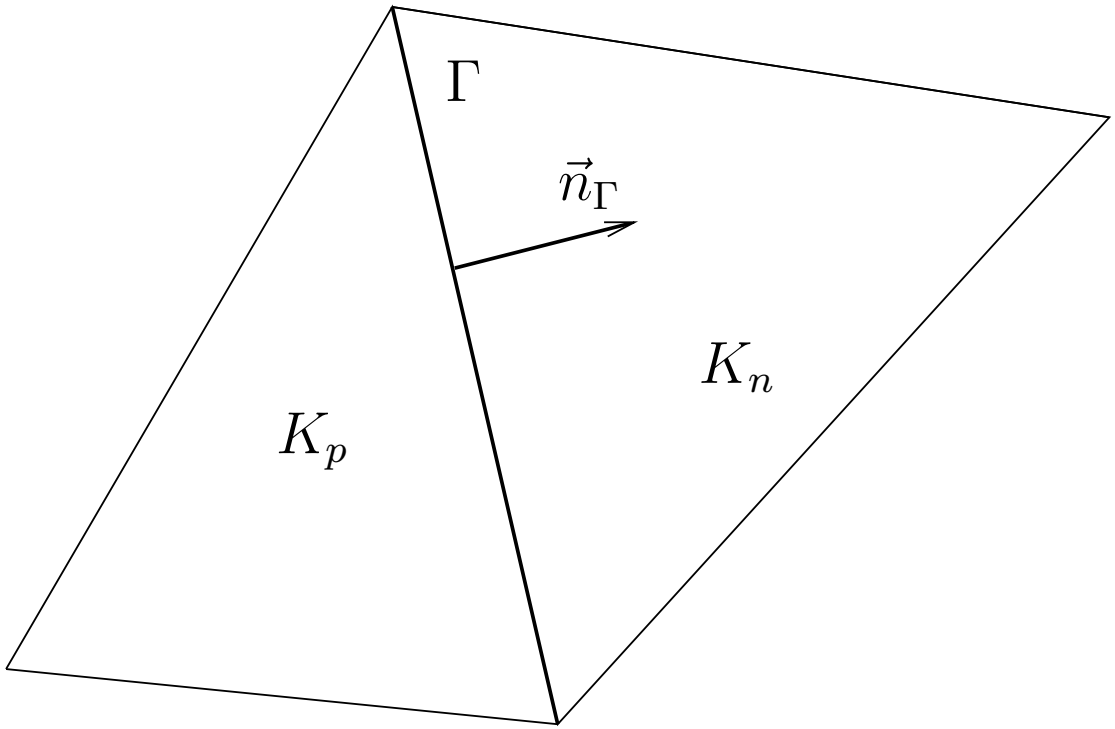
  - the space of piecewise polynomial functions

$$S_{hp} \equiv \{v; \; v \in L^2(\Omega), \; v|_K \in P_{p_K}(K) \; \forall K \in \mathcal{T}_h\} \subset H^1_w(\Omega, \mathcal{T}_h)$$

Example of a function from $S_{hp} \subset H^1_w(\Omega, \mathcal{T}_h)$



Notation - trace, mean value, jump



- $v|_\Gamma^{(p)} \equiv$ trace of $v|_{K_p}$ on Γ and $v|_\Gamma^{(n)} \equiv$ trace of $v|_{K_n}$ on Γ,
- $\langle v \rangle_\Gamma = \frac{1}{2}\left(v|_\Gamma^{(p)} + v|_\Gamma^{(n)}\right)$ and $[v]_\Gamma = v|_\Gamma^{(p)} - v|_\Gamma^{(n)}$,
- $\langle v \rangle_\Gamma \equiv [v]_\Gamma \equiv v|_\Gamma^{(p)}$, $\Gamma \subset \partial\Omega$.

Space semi-discretization

- let u be a strong (regular) solution on Ω,
- we multiply (2) by $v \in H^2_w(\Omega, \mathcal{T}_h)$,
- integrate over each $K \in \mathcal{T}_h$,
- apply Green's theorem,
- sum over all $K \in \mathcal{T}_h$,
- we include additional terms vanishing for regular solution,
- we obtain the identity

$$\left(\frac{\partial u}{\partial \hat{t}}(\hat{t}), v\right) + a_h(u(\hat{t}), v) + b_h(u(\hat{t}), v) + J_h(u(\hat{t}), v) \tag{4}$$
$$+\gamma(\hat{t})(u(\hat{t}), v) = \ell_h(v)(\hat{t}) \qquad \forall v \in H^2_w(\Omega, \mathcal{T}_h)\ \forall \hat{t} \in (0, T)$$

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h(u, v) + b_h(u, v) + J_h(u, v) + \gamma(u, v) = \ell_h(v)(\hat{t})$$

- time derivation, $u = u(t) \in H^2_w(\Omega, \mathcal{T}_h), \hat{t} \in (0, T)$,
- semi-implicit linearization of backward Euler method.

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h(u,v) + b_h(u,v) + J_h(u,v) + \gamma(u,v) = \ell_h(v)(\hat{t})$$

- diffusion terms:

$$\begin{aligned} a_h(u,v) &= \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{D}(x)\,\nabla u\cdot\nabla v\,\mathrm{d}x \\ &- \sum_{\Gamma\in\mathcal{F}_h^I\cup\mathcal{F}_h^D} \int_\Gamma \left\langle \boldsymbol{D}(x)\,\nabla u\cdot\vec{n}\right\rangle [v]\,\mathrm{d}S \\ &+ \sum_{\Gamma\in\mathcal{F}_h^I\cup\mathcal{F}_h^D} \int_\Gamma \left\langle \boldsymbol{D}(x)\,\nabla v\cdot\vec{n}\right\rangle [u]\,\mathrm{d}S, \end{aligned}$$

- $[u]_\Gamma = 0, \langle \boldsymbol{D}(x)\,\nabla u\rangle_\Gamma = \boldsymbol{D}(x)\,\nabla u|_\Gamma^{(p)} = \boldsymbol{D}(x)\,\nabla u|_\Gamma^{(n)}, \forall\,\Gamma\in\mathcal{F}_h^I$

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h(u,v) + b_h(u,v) + J_h(u,v) + \gamma(u,v) = \ell_h(v)(\hat{t})$$

- convection terms:

$$\begin{aligned} b_h(u,v) \;=\; & -\sum_{K\in\mathcal{T}_h}\int_K \vec{a}(x,\hat{t})u\cdot\nabla v\,\mathrm{d}x \\ & +\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma H\left(u|_\Gamma^{(p)}, u|_\Gamma^{(n)}, \vec{n}_\Gamma\right)[v]\,\mathrm{d}S \\ & +\sum_{\Gamma\in\mathcal{F}_h\setminus\mathcal{F}_h^I}\int_\Gamma H\left(u|_\Gamma^{(p)}, u^*|_\Gamma, \vec{n}_\Gamma\right)[v]\,\mathrm{d}S \end{aligned}$$

- physical flux $\Leftrightarrow$ numerical flux:

$$\vec{a}(x,,\hat{t})u\cdot\vec{n}|_\Gamma \approx H\left(u|_\Gamma^{(p)}, u|_\Gamma^{(n)}, \vec{n}_\Gamma\right), \quad \Gamma\subset K_p\cap K_n.$$

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h(u,v) + b_h(u,v) + J_h(u,v) + \gamma(u,v) = \ell_h(v)(\hat{t})$$

- numerical flux (concept of upwinding):

$$H\left(u|_\Gamma^{(p)}, u|_\Gamma^{(n)}, \vec{n}_\Gamma\right) = \begin{cases} \vec{a}(x,\hat{t})u|_\Gamma^{(p)} \cdot \vec{n}_\Gamma, & \text{if } E > 0 \\ \vec{a}(x,\hat{t})u|_\Gamma^{(n)} \cdot \vec{n}_\Gamma, & \text{if } E \leq 0 \end{cases}$$

where $E = \sum_{i=1}^2 a_i(x,\hat{t})n_i$, $\vec{a}(x,\hat{t}) = \big(a_1(x,\hat{t}), a_2(x,\hat{t})\big)$ with

$$a_k(x,\hat{t}) = \left(\sigma_k^2 + \frac{1}{2}\rho\sigma_1\sigma_2 - r + \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T - \hat{t}}\right) x_k, \quad k = 1,2,$$

- $u^*|_\Gamma$ is chosen according to boundary conditions.

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h(u,v) + b_h(u,v) + J_h(u,v) + \gamma(u,v) = \ell_h(v)(\hat{t})$$

- interior and boundary penalty:

$$J_h(u,v) = \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma \frac{\omega}{|\Gamma|} [u]\,[v]\,\mathrm{d}S + \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \frac{\omega}{|\Gamma|} u\,v\,\mathrm{d}S,$$

- $|\Gamma|$ is the length of edge Γ,
- weighted parameter $\omega = \frac{\sigma_{min}^2}{4}(1 - |\rho|)$,
- u is regular $\Rightarrow [u]_\Gamma = 0 \Rightarrow \int_\Gamma [u]\,[v]\,\mathrm{d}S = 0, \quad \forall\, \Gamma \in \mathcal{F}_h^I$.

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h(u,v) + b_h(u,v) + J_h(u,v) + \gamma(u,v) = \ell_h(v)(\hat{t})$$

- reaction terms with factor:

$$\gamma(x,\hat{t}) = 3r - \frac{4\alpha_1 x_1 + 4\alpha_2 x_2 - 3}{T - \hat{t}} - \sigma_1^2 - \sigma_2^2 - \rho\sigma_1\sigma_2$$

- (broken) $L^2(\Omega)$-inner product

$$(u,v) = \int_\Omega uv\,\mathrm{d}x = \sum_{K\in\mathcal{T}_h} \int_K uv\,\mathrm{d}x$$

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h(u,v) + b_h(u,v) + J_h(u,v) + \gamma(u,v) = \ell_h(v)(\hat{t})$$

- right-hand-side $\Leftrightarrow$ boundary conditions

$$\begin{aligned}\ell_h(v)(\hat{t}) &= \sum_{\Gamma\in\mathcal{F}_h^D} \int_\Gamma \boldsymbol{D}(x)\,\nabla v\cdot\vec{n}_\Gamma\, u^*(\hat{t})\mathrm{d}S \\ &+ \sum_{\Gamma\in\mathcal{F}_h^D} \int_\Gamma \frac{\omega}{|\Gamma|}\, u^*(\hat{t})\, v\mathrm{d}S\end{aligned}$$

- terms from homogeneous Neumann b.c. vanish.

Semi-discrete DG scheme

- $S_{hp} \subset H^2_w(\Omega, \mathcal{T}_h) \Rightarrow (4)$ makes sense for $u_h, v_h \in S_{hp}$
- we define new bilinear form

$$\mathcal{C}_h(u, v) := a_h(u, v) + b_h(u, v) + J_h(u, v) + \gamma(u, v)$$

Semi-discrete DG solution

We say that u_h is a semi-discrete DG solution iff

$$\begin{aligned}
&\text{a)} \quad u_h \in C^1(0, T; S_{hp}), \\
&\text{b)} \quad \left(\frac{\partial u_h(\hat{t})}{\partial \hat{t}}, v_h\right) + \mathcal{C}_h(u_h(\hat{t}), v_h) = 0 \qquad (5) \\
&\qquad\qquad \forall\, v_h \in S_{hp},\ \hat{t} \in (0, T) \\
&\text{c)} \quad u_h(0) = u_h^0,\ u_h^0 \text{ is } S_{hp}\text{-approximation of } u^0
\end{aligned}$$

- semi-discrete problem (5) represents ODEs with IC,

Existence and uniqueness of semi-discrete solution

Theorem

Problem (5) has a unique semi-discrete solution.

Proof (sketch 1/2)

Let $\{v_j\}_{j=1}^{DOF}$ be the standard base of S_{hp}, i.e.

$$u_h(x,\hat{t}) = \sum_{j=1}^{DOF} \xi_j(\hat{t}) v_j(x).$$

We substitute this into (5), choose and set

$$G(\hat{t}) = (\xi_1(\hat{t}), \dots, \xi_{DOF}(\hat{t}))^T,$$
$$L(\hat{t}) = (\ell_h(v_1)(\hat{t}), \dots, \ell_h(v_{DOF})(\hat{t}))^T,$$
$$\mathbf{C}(\hat{t}) = (c_{ij}(\hat{t}))_{DOF\times DOF},\ c_{ij}(\hat{t}) = \mathcal{C}_h(v_j, v_i).$$

Existence and uniqueness of semi-discrete solution

Theorem

Problem (5) has a unique semi-discrete solution.

Proof (sketch 2/2)

We rewrite (5) as follows: find $G(\hat{t})$, $\forall \hat{t} \in (0, T)$ such that

$$G'(\hat{t}) = -\mathbf{C}(\hat{t}) \cdot G(\hat{t}) + L(\hat{t}),$$
$$G(0) = (\xi_1(0), \ldots, \xi_{DOF}(0))^T \quad \text{(given by } u_h(0)\text{)}.$$

Numerical flux H is Lipschitz continuous and form $\mathcal{C}_h(\cdot, \cdot)$ is bilinear $\Rightarrow$ RHS of afore-mentioned ODEs is Lipschitz continuous. The proof is completed by theory of ODEs.

Time discretization

- linearity of form $\mathcal{C}_h(\cdot,\cdot) \Rightarrow$ implicit treatment,
- implicit approach via backward Euler method
- let $0 = t_0 < t_1 < \cdots < t_r = T$ be a partition of $(0, T)$ with constant step $\tau = T/r$, $u_h(t_m) \approx u_h^m \in S_{hp}, \quad m = 0, \ldots, r$

First order implicit scheme

$$\left(\frac{1}{\tau} + \gamma\right)(u_h^{m+1}, v_h) + a_h(u_h^{m+1}, v_h) + b_h(u_h^{m+1}, v_h) \tag{6}$$
$$+ J_h(u_h^{m+1}, v_h) = \frac{1}{\tau}(u_h^m, v_h) + \ell_h(v_h)(t_{m+1}) \quad \forall\, v_h \in S_{hp},$$

- problem (6) $\Longleftrightarrow$ system of linear algebraic equations.

Existence and uniqueness of fully discrete solution

Theorem

Problem (6) has a unique discrete solution.

Proof (sketch 1/3)

Suppose $u_h^m \equiv 0$, i.e. $\ell_h(v_h) = 0$, then problem (6) is equivalent to

$$\left(\frac{1}{\tau}+\gamma\right)(u_h^{m+1}, v_h)+a_h(u_h^{m+1}, v_h)+b_h(u_h^{m+1}, v_h)+J_h(u_h^{m+1}, v_h) = 0$$

Problem (6) is linear algebraic system $\Rightarrow$ the existence is implied by uniqueness.

Existence and uniqueness of fully discrete solution

Theorem

Problem (6) has a unique discrete solution.

Proof (sketch 2/3)

Taking $v_h = u_h^{m+1}$ and neglecting some positive terms, we obtain

$$\frac{1}{\tau}\|u_h^{m+1}\|^2 \leq |b_h(u_h^{m+1}, u_h^{m+1})| + |\gamma|\|u_h^{m+1}\|^2$$

Since

$$|b_h(u_h^{m+1}, u_h^{m+1})| \leq C\left(J_h(u_h^{m+1}, u_h^{m+1})^{1/2} + |u_h^{m+1}|_{1,w}\right)\|u_h^{m+1}\|$$

then

$$\begin{aligned}\|u_h^{m+1}\|^2 \quad &\leq \quad \tau C\left(J_h(u_h^{m+1}, u_h^{m+1})^{1/2} + |u_h^{m+1}|_{1,w}\right)\|u_h^{m+1}\| \\ &\quad + \tau|\gamma|\|u_h^{m+1}\|^2\end{aligned}$$

Existence and uniqueness of fully discrete solution

Theorem

Problem (6) has a unique discrete solution.

Proof (sketch 3/3)

For sufficiently small τ, it holds

$$\|u_h^{m+1}\|^2 \leq \delta \|u_h^{m+1}\|^2, \quad \delta \in (0,1)$$

Then $\|u_h^{m+1}\| = 0$, i.e. $u_h^{m+1} \equiv 0$.
We prove that exists only trivial solution for homogeneous linear algebraic system $\Rightarrow$ existence and uniqueness.

Linear algebraic problem

- standard basis of space S_{hp}: $\{v_j\}_{j=1}^{DOF}$,
- linear algebraic representation:

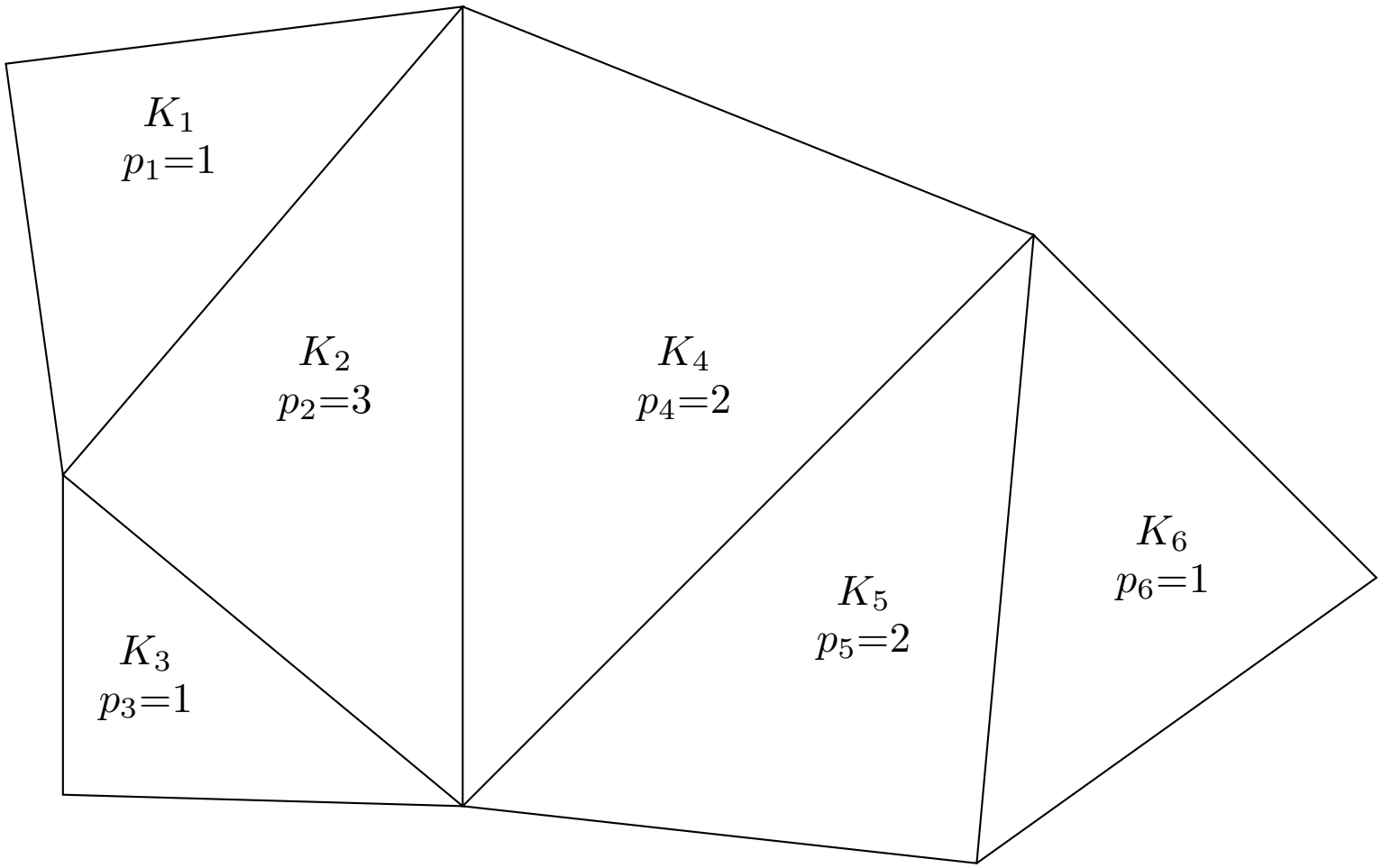
$$\sum_{j=1}^{DOF} \xi_j^m v_j(x) = u_h^m(x) \longleftrightarrow U_m = (\xi_1^m, \ldots, \xi_{DOF}^m)^T \in \boldsymbol{R}^{DOF}$$

- sparse matrix equation

$$\underbrace{\Big((1/\tau + \gamma)\mathbf{M} + \mathbf{A} + \mathbf{B} + \mathbf{J}\Big)}_{\mathbf{C}} U_{m+1} = 1/\tau\, \mathbf{M} U_m + F_{m+1} \qquad (7)$$

- U_m unknown vector, $\mathbf{M}$ mass matrix, $\mathbf{A}$ stiffness matrix,
- $\mathbf{B}$ matrix representing the form b_h, $\mathbf{J}$ penalty matrix,
- $F_{m+1} = \ell_h(v_h)(t_{m+1})$ right-hand side vector.

Fictional triangulation



Structure of system matrix $\mathbf{C}$

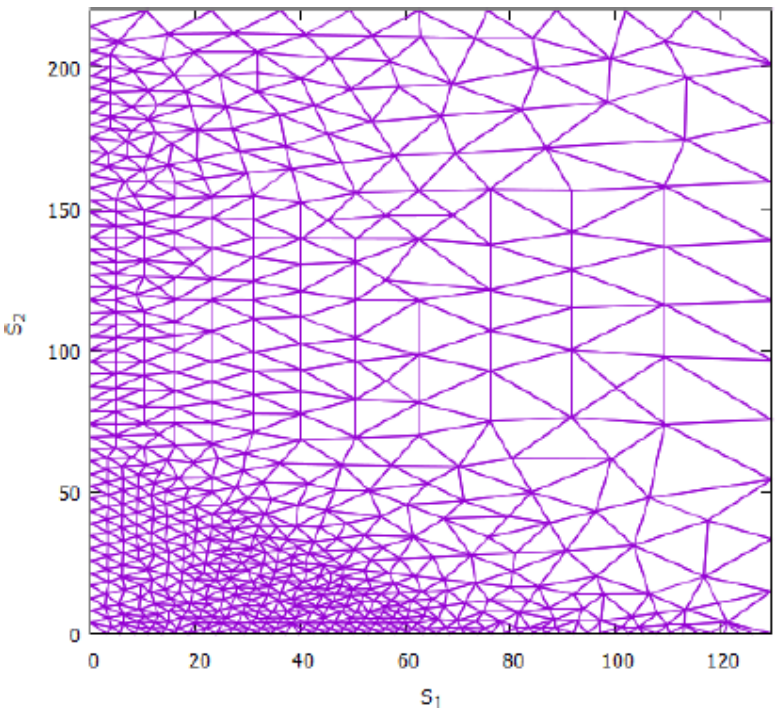
$$
\begin{pmatrix}
\underset{12\times 12}{\boldsymbol{C}_{k,1,1}} & \underset{12\times 40}{\boldsymbol{C}_{k,1,2}} & & & & \\
\underset{40\times 12}{\boldsymbol{C}_{k,2,1}} & \underset{40\times 40}{\boldsymbol{C}_{k,2,2}} & \underset{40\times 12}{\boldsymbol{C}_{k,2,3}} & \underset{40\times 24}{\boldsymbol{C}_{k,2,4}} & & \\
 & \underset{12\times 40}{\boldsymbol{C}_{k,3,2}} & \underset{12\times 12}{\boldsymbol{C}_{k,3,3}} & & & \\
 & \underset{24\times 40}{\boldsymbol{C}_{k,4,2}} & & \underset{24\times 24}{\boldsymbol{C}_{k,4,4}} & \underset{24\times 24}{\boldsymbol{C}_{k,4,5}} & \\
 & & & \underset{24\times 24}{\boldsymbol{C}_{k,5,4}} & \underset{24\times 24}{\boldsymbol{C}_{k,5,5}} & \underset{24\times 12}{\boldsymbol{C}_{k,5,6}} \\
 & & & & \underset{12\times 24}{\boldsymbol{C}_{k,6,5}} & \underset{12\times 12}{\boldsymbol{C}_{k,6,6}}
\end{pmatrix}
$$

Freefem++ solver

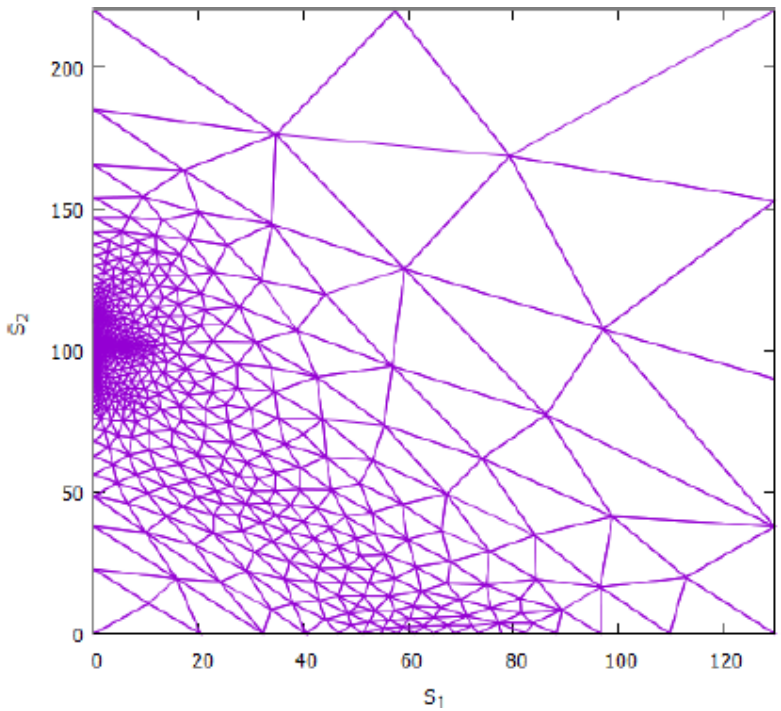
- source: http://www.freefem.org/ff++/,
- C++ code and software to solve PDEs numerically,
- user friendly input language allows for a quick specification of any PDE based on a variational formulation,
- easy mesh generation/adaptation,
- available library of a wide range of finite elements,
- implemented sparse solvers,
- basic post-processing/vizualization,
- for more details see [Hecht].

Implementation settings

- adaptively refined grid,
- constant time step $\tau = 1/365$ (i.e. 1 day),
- piecewise P_1 (linear) approximations, sparse solver GMRES



initial



adapted

Parameter settings

- basket put option with 60% Allianz ($\alpha_1 = 0.6$) and 40% Deutshce Bank ($\alpha_2 = 0.4$),
- current date: September 13, 2011 (Tuesday),
- volatilities σ_1 and σ_2 : 4 ways of its determination,
- expiration date $T = 0.257534$ (i.e. 94 days),
- risk-free interest rate $r = 0.01557$ p.a.,
- Pearson linear correlation $\rho = 0.88$,
- two stocks trades at $S_1^{ref} = 59.79$ and $S_2^{ref} = 23.40$,
- reference point $[S_1^{ref}, S_2^{ref}, A^{ref}]$, where $A^{ref} = \alpha_1 S_1^{ref} + \alpha_2 S_2^{ref}$
- maximal stock prices $x_1^{max} = 5.0$ and $x_2^{max} = 7.0$.

Determination of volatility

Real market data of individual options:

(A) constant volatilities $\sigma_1 = 0.6392, \sigma_2 = 0.9461$,

(B) piecewise constant implied volatility as a function of S_i, i.e. $\sigma_i = \sigma_i(S_i),\ i = 1, 2$,

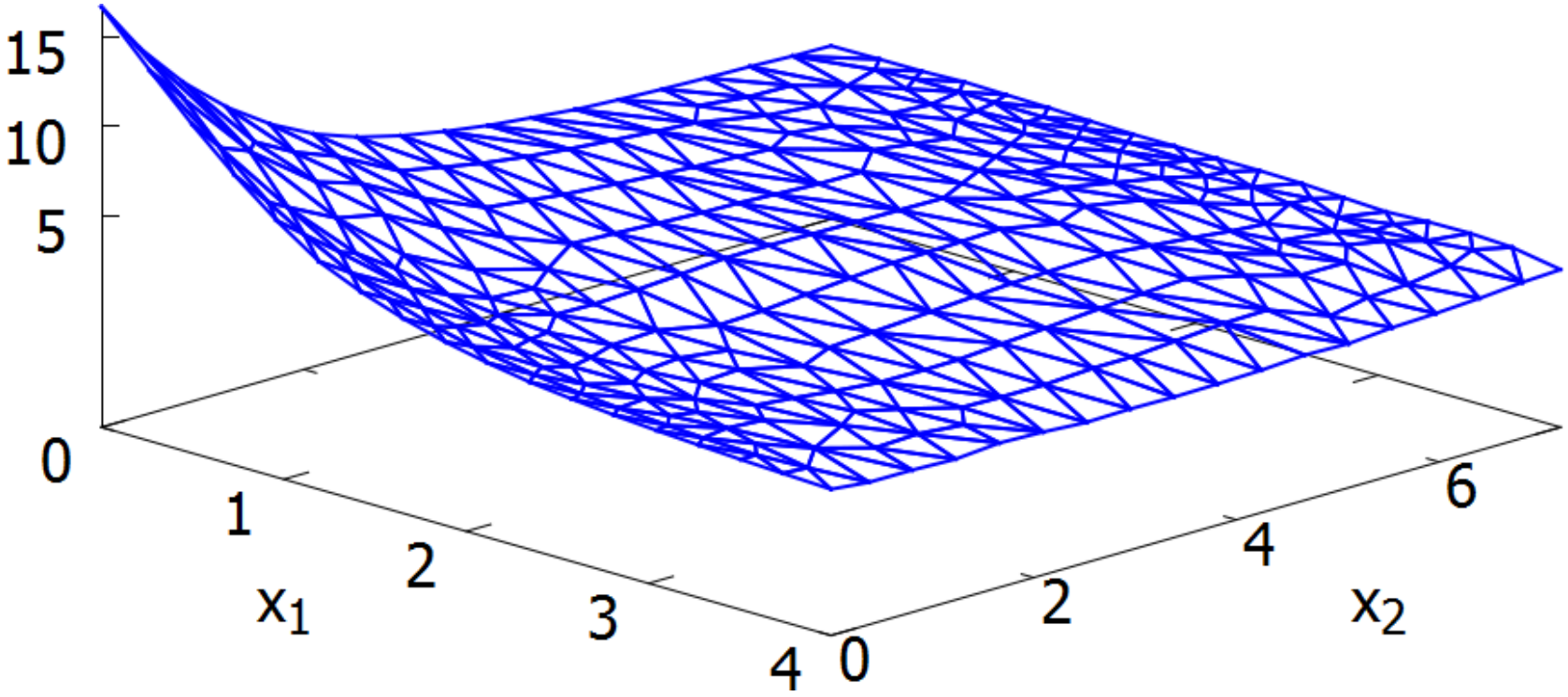
(C) piecewise constant implied volatility as a function of $\alpha_i S_i$, i.e. $\sigma_i = \sigma_i(\alpha_i S_i),\ i = 1, 2$,

(D) piecewise constant implied volatility as a function of moneyness, i.e.

$$\sigma_i = \sigma_i \left(\frac{\alpha_i S_i}{\alpha_1 S_1^{ref} + \alpha_2 S_2^{ref}} \exp(-rT) \right),\ i = 1, 2.$$

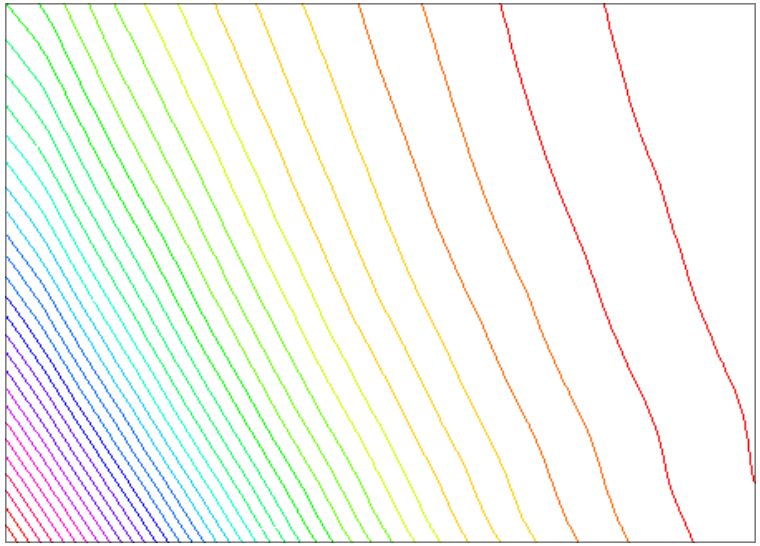
Asian basket with floating strike (1)

- space-time plot of solution with approach (D)



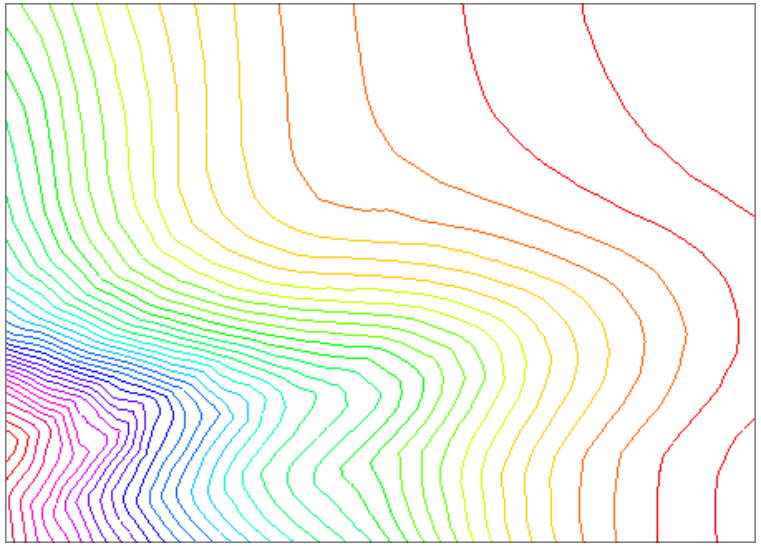
Asian basket with floating strike (2)

- comparison of approaches, isolvalues of solutions, zoom on $(0, 2.50) \times (0, 1.80)$



(A)

4.08903 (Asian basket)
6.15615 (basket)



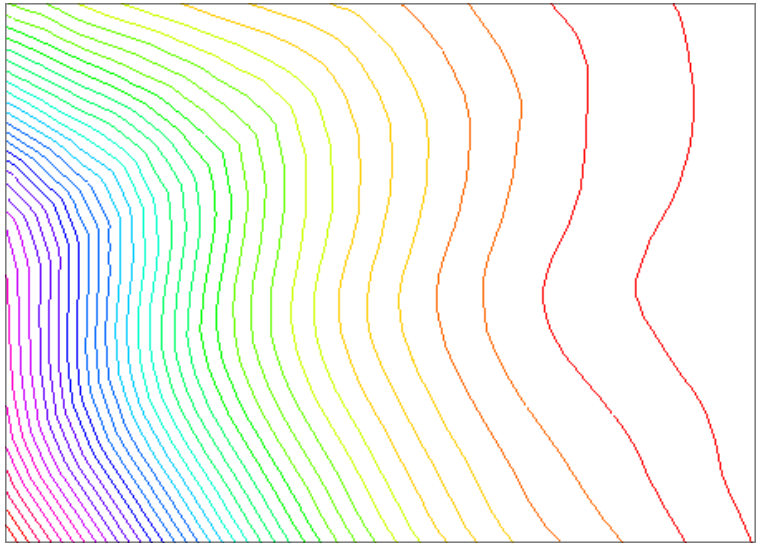
(B)

6.70615 (Asian basket)
8.55314 (basket)

- $\#\mathcal{T}_h \approx 4\,000$, results at the reference node $[1.3218, 0.5173]$

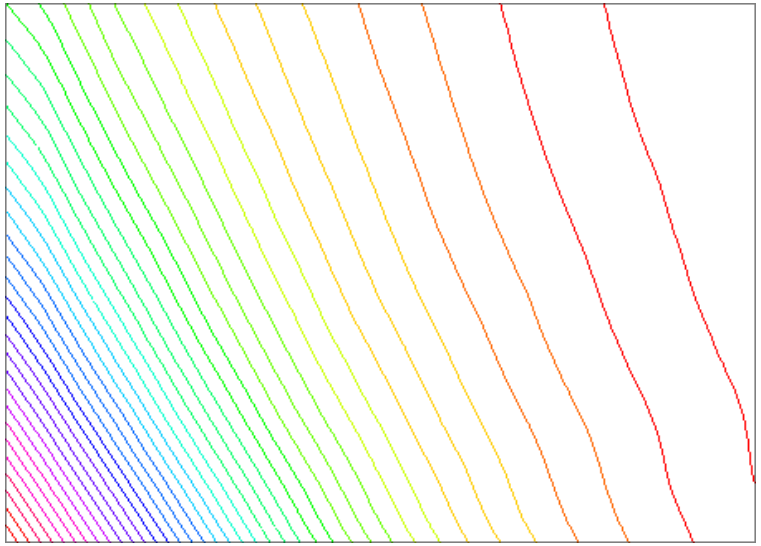
Asian basket with floating strike (3)

- comparison of approaches, isolvalues of solutions, zoom on $(0, 2.50) \times (0, 1.80)$



(C)

4.08995 (Asian basket)
5.52614 (basket)



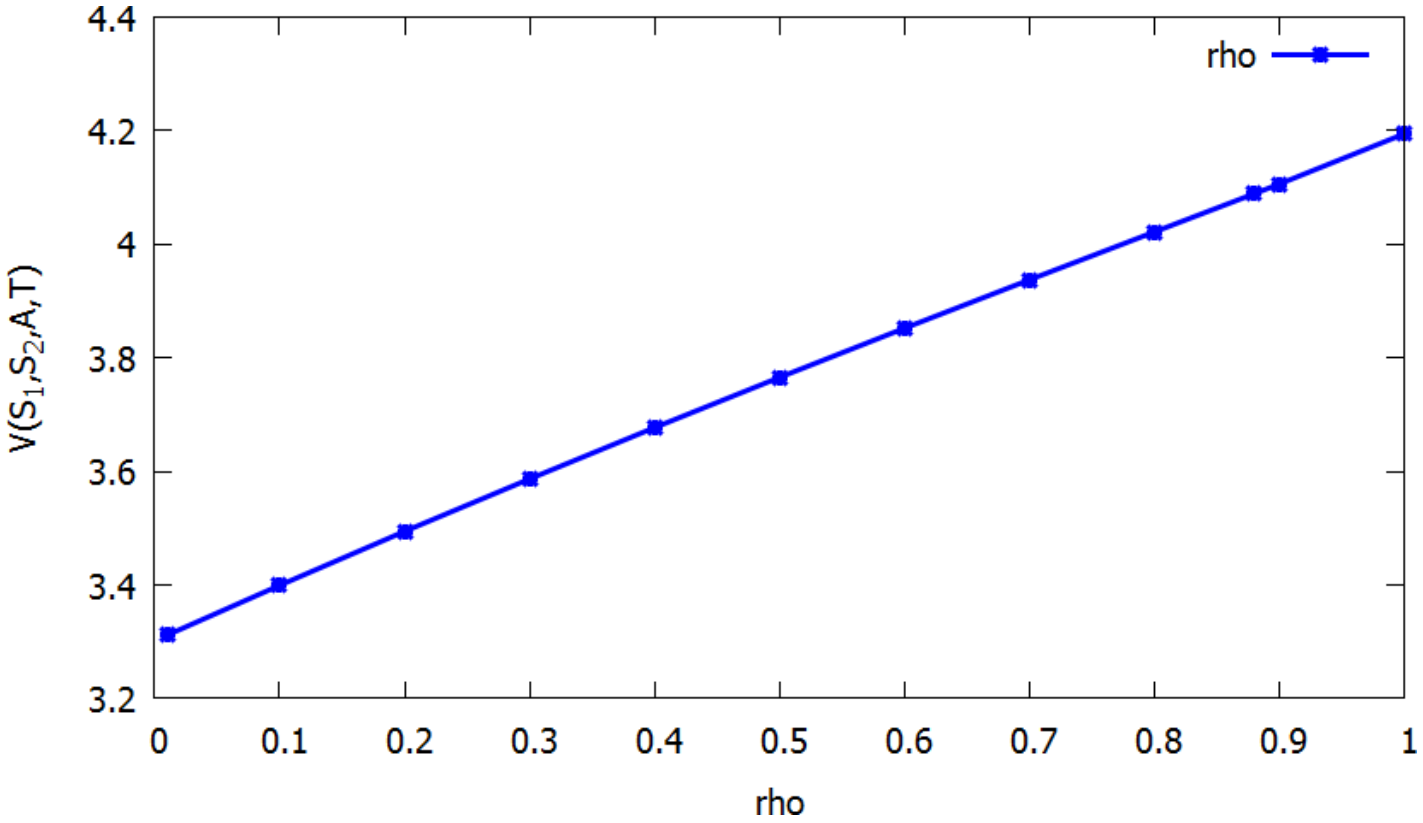
(D)

4.08913 (Asian basket)
5.70892 (basket)

- $\#\mathcal{T}_h \approx 4\,000$, results at the reference node $[1.3218, 0.5173]$

Asian basket with floating strike (4)

- dependency among particular risk sources,
- basket option value at ref. node vs. correlation plot,
- approach (D) for volatilities



1 Motivation

2 Path-dependent multi-asset options

3 Asian two-asset option with floating strike

4 Asian two-asset option with fixed strike

3D PDE model

- continuous arithmetic Asian two-asset basket option with floating strike = the same PDE model as for Asian two-asset basket option with fixed strike,
- for simplicity, we consider the case of puts only (the case of calls can be easy derived from put-call parity),
- different payoffs (puts): $V(S, A, T) = \max(K - A, 0)$
- different boundary conditions at far-field boundary (puts)
  - no dependency on S_1 and S_2 for payoffs, i.e.,

$$\lim_{S_1 \to \infty} \frac{\partial V}{\partial S_1}(S_1, S_2, A, t) = 0, \quad \lim_{S_2 \to \infty} \frac{\partial V}{\partial S_2}(S_1, S_2, A, t) = 0,$$

  - zero price for large value of A, i.e.,

$$\lim_{A \to \infty} V(S_1, S_2, A, t) = 0$$

Dimensional reduction

- to avoid the higher complexity of 3D model
- change of variables (approach from [Rogers, Shi])

$$x_1 = \frac{K - A(T-t)/T}{S_1}, \quad x_2 = \frac{K - A(T-t)/T}{S_2}, \quad \hat{t} = T - t$$

- transformation of price function

$$u(x, \hat{t}) = \frac{1}{\sqrt{S_1 S_2}} V(S_1, S_2, A, t), \quad \text{where } x = [x_1, x_2]$$

- transformation of payoffs (= initial conditions)

$$u(x, 0) = \frac{1}{\sqrt{S_1 S_2}} V(S_1, S_2, A, T) = \max(\sqrt{x_1 x_2} \cdot \operatorname{sgn}(x_1), 0)$$

- transformation of PDE: homogeneity w.r.t. $\sqrt{S_1 S_2}$
  $\longrightarrow$ separation of factor $\sqrt{S_1 S_2}$ $\longrightarrow$ dimensional reduction from 3D to 2D

2D reformulation

- localization on a truncated computational domain Ω,
- convection-diffusion-reaction equation in divergence-free form

We seek $u : Q_T = \Omega \times (0, T) \to \boldsymbol{R}$ such that

$$\frac{\partial u}{\partial \hat{t}} - \operatorname{div}\left(\boldsymbol{D}(x) \cdot \nabla u\right) + \nabla \cdot \vec{f}(x, u) + \gamma(x) u = 0 \quad \text{in } Q_T, \tag{8}$$

$$B(u) = 0 \quad \text{on } \partial\Omega,$$

$$u(x, 0) = u^0(x), \quad x \in \Omega,$$

where

$$\boldsymbol{D}(x) \equiv \{D(x)_{kl}\}_{k,l=1}^{2} = \frac{1}{2}\begin{pmatrix} \sigma_1^2 x_1^2 & \rho\sigma_1\sigma_2 x_1 x_2 \\ \rho\sigma_1\sigma_2 x_1 x_2 & \sigma_2^2 x_2^2 \end{pmatrix},$$

2D reformulation

- localization on a truncated computational domain Ω,
- convection-diffusion-reaction equation in divergence-free form

We seek $u : Q_T = \Omega \times (0, T) \to \boldsymbol{R}$ such that

$$\frac{\partial u}{\partial \hat{t}} - \operatorname{div}(\boldsymbol{D}(x) \cdot \nabla u) + \nabla \cdot \vec{f}(x, u) + \gamma(x) u = 0 \quad \text{in } Q_T, \qquad (8)$$

$$B(u) = 0 \quad \text{on } \partial\Omega,$$

$$u(x, 0) = u^0(x), \quad x \in \Omega,$$

where $\vec{f}(x, u) = (f_1(x, u), f_2(x, u))^T$ with

$$f_k(x, u) = \left(\frac{\sigma_k^2}{2} + \rho\sigma_1\sigma_2 + r + \frac{\alpha_1}{Tx_1} + \frac{\alpha_2}{Tx_2} \right) x_k \cdot u, \; k = 1, 2$$

2D reformulation

- localization on a truncated computational domain Ω,
- convection-diffusion-reaction equation in divergence-free form

We seek $u : Q_T = \Omega \times (0, T) \to \boldsymbol{R}$ such that

$$\frac{\partial u}{\partial \hat{t}} - \operatorname{div}(\boldsymbol{D}(x) \cdot \nabla u) + \nabla \cdot \vec{f}(x, u) + \gamma(x) u = 0 \quad \text{in } Q_T, \qquad (8)$$

$$B(u) = 0 \quad \text{on } \partial\Omega,$$

$$u(x, 0) = u^0(x), \quad x \in \Omega,$$

where

$$\gamma(x) = -2r - \frac{\alpha_1}{Tx_1} - \frac{\alpha_2}{Tx_2} - \frac{3}{8}\sigma_1^2 - \frac{3}{8}\sigma_2^2 - \frac{9}{4}\rho\sigma_1\sigma_2$$

2D reformulation

- localization on a truncated computational domain Ω,
- convection-diffusion-reaction equation in divergence-free form

We seek $u : Q_T = \Omega \times (0, T) \to \boldsymbol{R}$ such that

$$\frac{\partial u}{\partial \hat{t}} - \operatorname{div}(\boldsymbol{D}(x) \cdot \nabla u) + \nabla \cdot \vec{f}(x, u) + \gamma(x) u = 0 \quad \text{in } Q_T, \qquad (8)$$

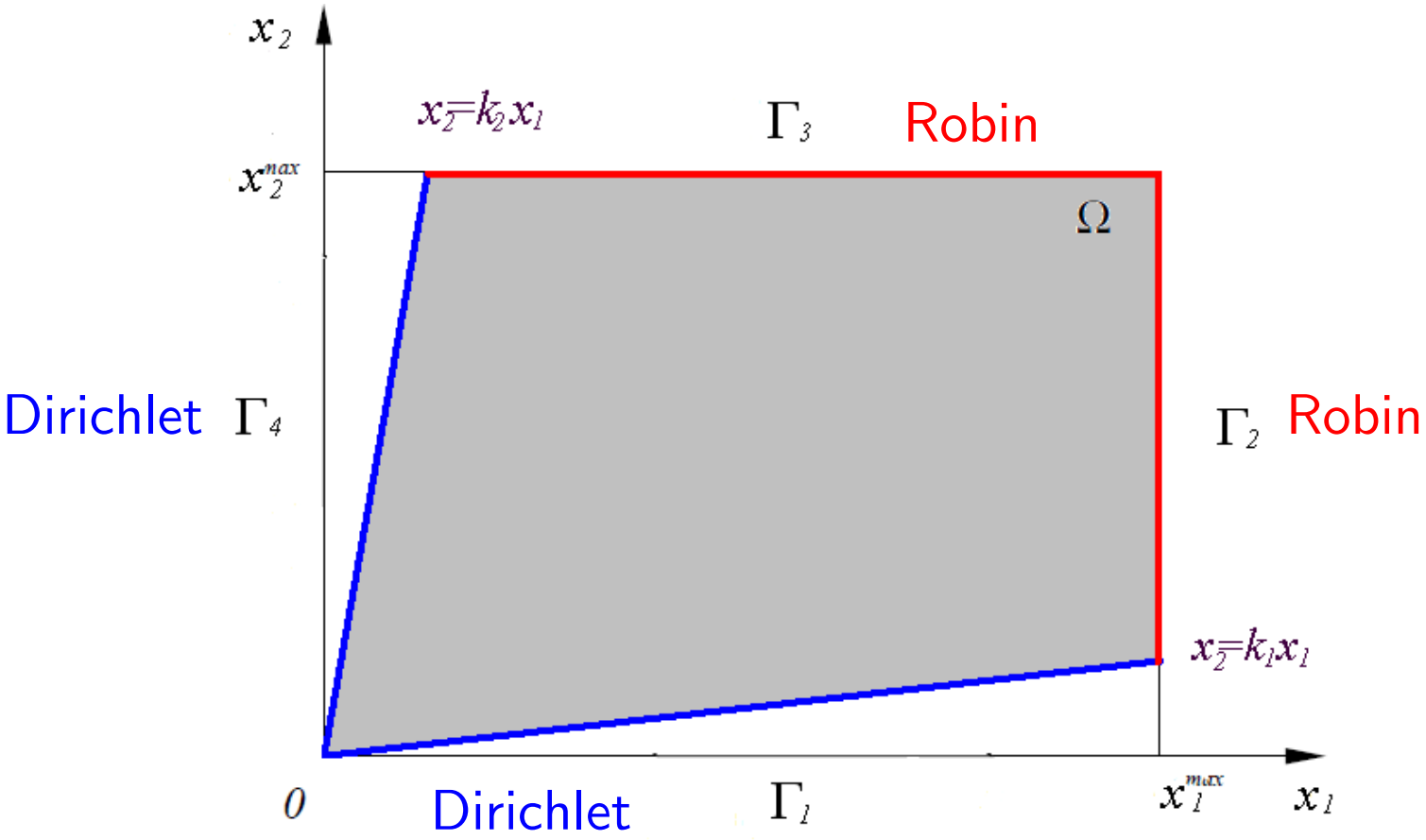
$$B(u) = 0 \quad \text{on } \partial\Omega,$$

$$u(x, 0) = u^0(x), \quad x \in \Omega,$$

- $\hat{t}$ - time to maturity $\longrightarrow$ i.c. given by payoffs u^0
- boundary conditions $B(u)$ specified latter

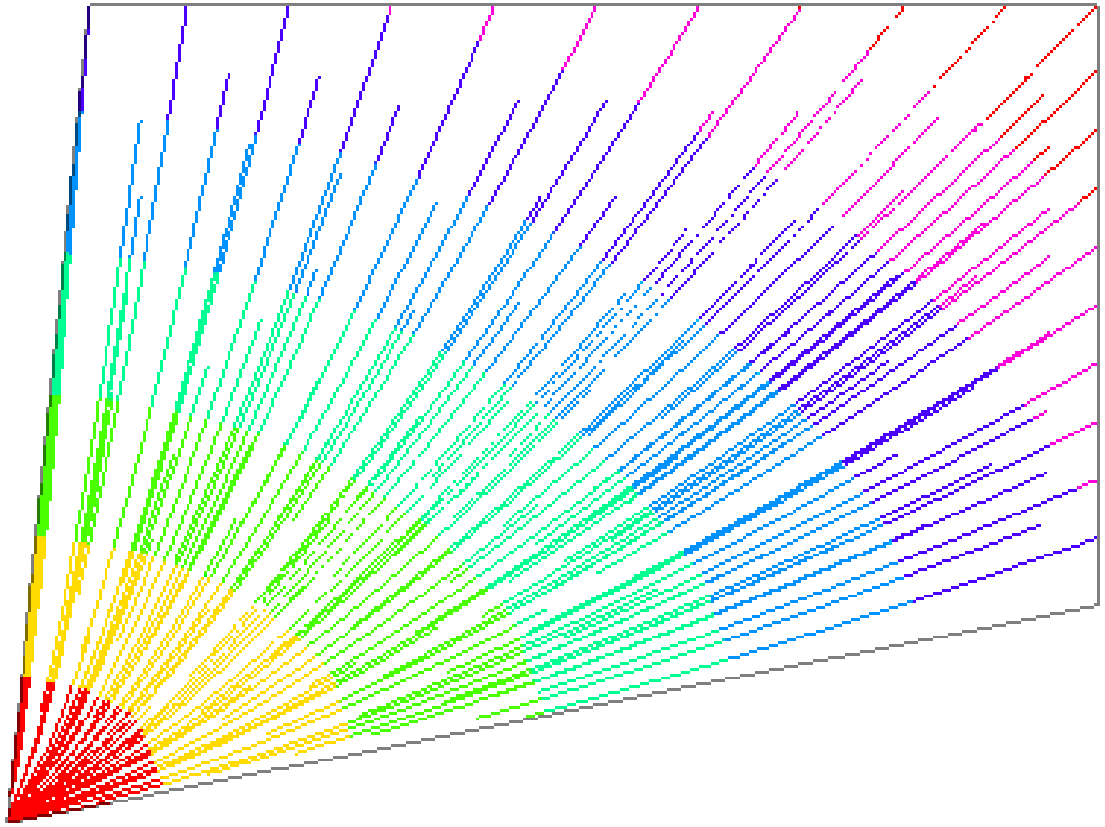
Boundary conditions (1)

- $x_1 \neq 0$ and $x_2 \neq 0 \longrightarrow$ domain Ω is a convex quadrilateral



Boundary conditions (2)

- setting boundary conditions in consistency with the vector field induced by physical fluxes, i.e. $\left(\frac{\partial f_1}{\partial u}, \frac{\partial f_2}{\partial u}\right)$



Boundary conditions (3)

- reformulation of original b.c.
- non-homogeneous Dirichlet b.c. on axes $x_1 = k_1 x_2$ and $x_2 = k_2 x_2$ $(0 < k_1 < k_2)$

$$u(x_1, x_2, \hat{t}) = u_D(x_1, x_2, \hat{t}), \quad \text{on } \Gamma_i, i = 1, 4, \ \hat{t} > 0,$$

  where u_D is the solution of 1D problem on $\Gamma_i, i = 1, 4$,
- Robin b.c. at infinity

$$\frac{1}{2} u(x_1, x_2, \hat{t}) - x_i \frac{\partial u}{\partial x_i}(x_1, x_2, \hat{t}) = 0 \quad \text{on } \Gamma_i, i = 2, 3, \ \hat{t} > 0.$$

Weak formulation and DG discretization

- standard Sobolev and Lebesgue spaces (no weighted spaces),
- existence and uniqueness of weak solution

$$\left(\frac{\partial u}{\partial \hat{t}}, v\right) + (\mathcal{L}_t u, v) = 0 \quad \forall\, v \in H_0^1,\ \text{a.e. } \hat{t} \in (0, T),$$

- DG approach analogous to Asian basket with floating strike,
- similar numerical scheme

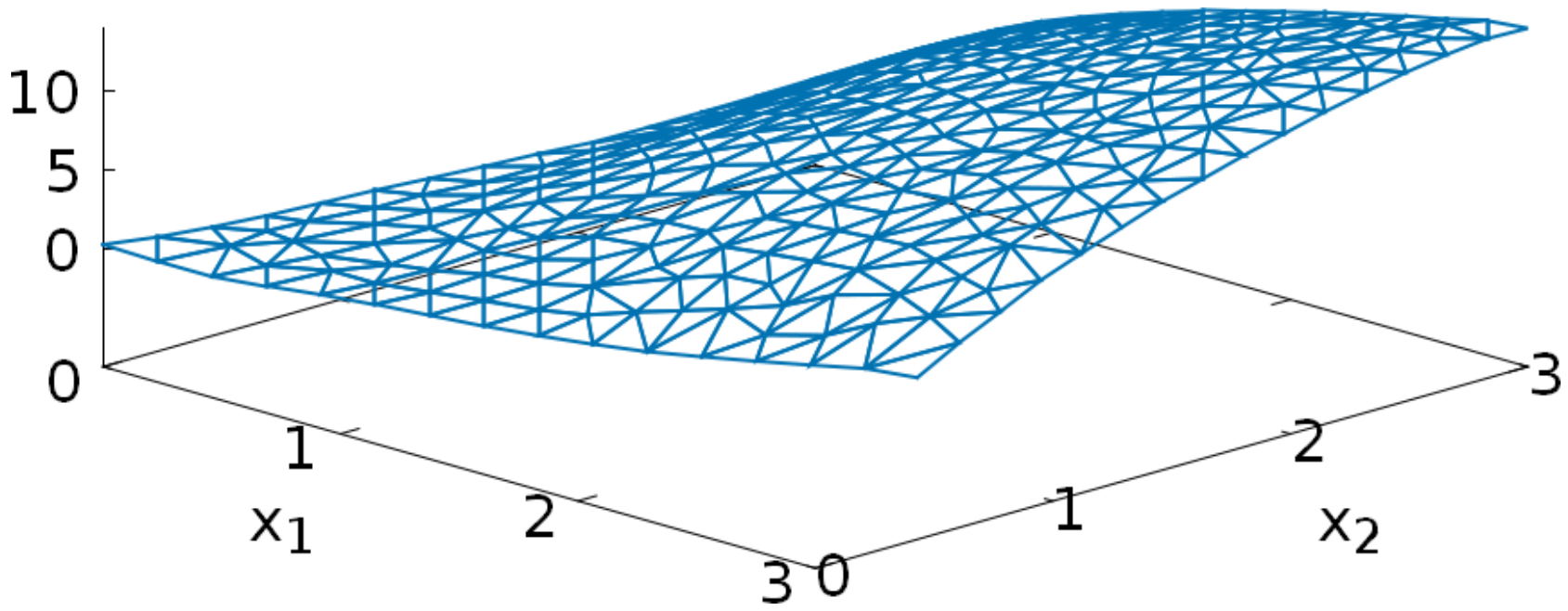
$$\left(\frac{1}{\tau} + \gamma\right)(u_h^{m+1}, v_h) + a_h(u_h^{m+1}, v_h) + b_h(u_h^{m+1}, v_h) + J_h(u_h^{m+1}, v_h) = \frac{1}{\tau}(u_h^m, v_h) + \ell_h(v_h)(t_{m+1}) \quad \forall\, v_h \in S_{hp}$$

Parameter settings

- basket put option with 60% Allianz ($\alpha_1 = 0.6$) and 40% Deutshce Bank ($\alpha_2 = 0.4$),
- strike at 40 Euro,
- current date: September 13, 2011 (Tuesday),
- volatilities: $\sigma_1 = 0.6392$ and $\sigma_2 = 0.9461$,
- expiration date $T = 0.257534$ (i.e. 94 days),
- risk-free interest rate $r = 0.01557$ p.a.,
- Pearson linear correlation $\rho = 0.88$,
- two stocks trades at $S_1^{ref} = 59.79$ and $S_2^{ref} = 23.40$,
- reference point $[S_1^{ref}, S_2^{ref}, A^{ref}]$, where $A^{ref} = \alpha_1 S_1^{ref} + \alpha_2 S_2^{ref}$
- maximal values $x_1^{max} = 3.0$ and $x_2^{max} = 3.0$.

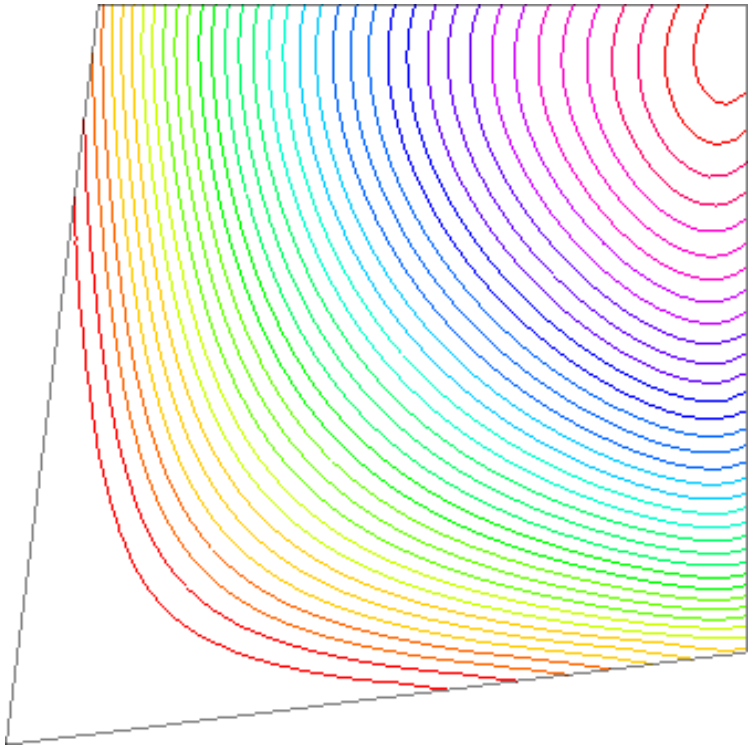
Asian basket with fixed strike (1)

- space-time plot of solution at maturity)



Asian basket with fixed strike (2)

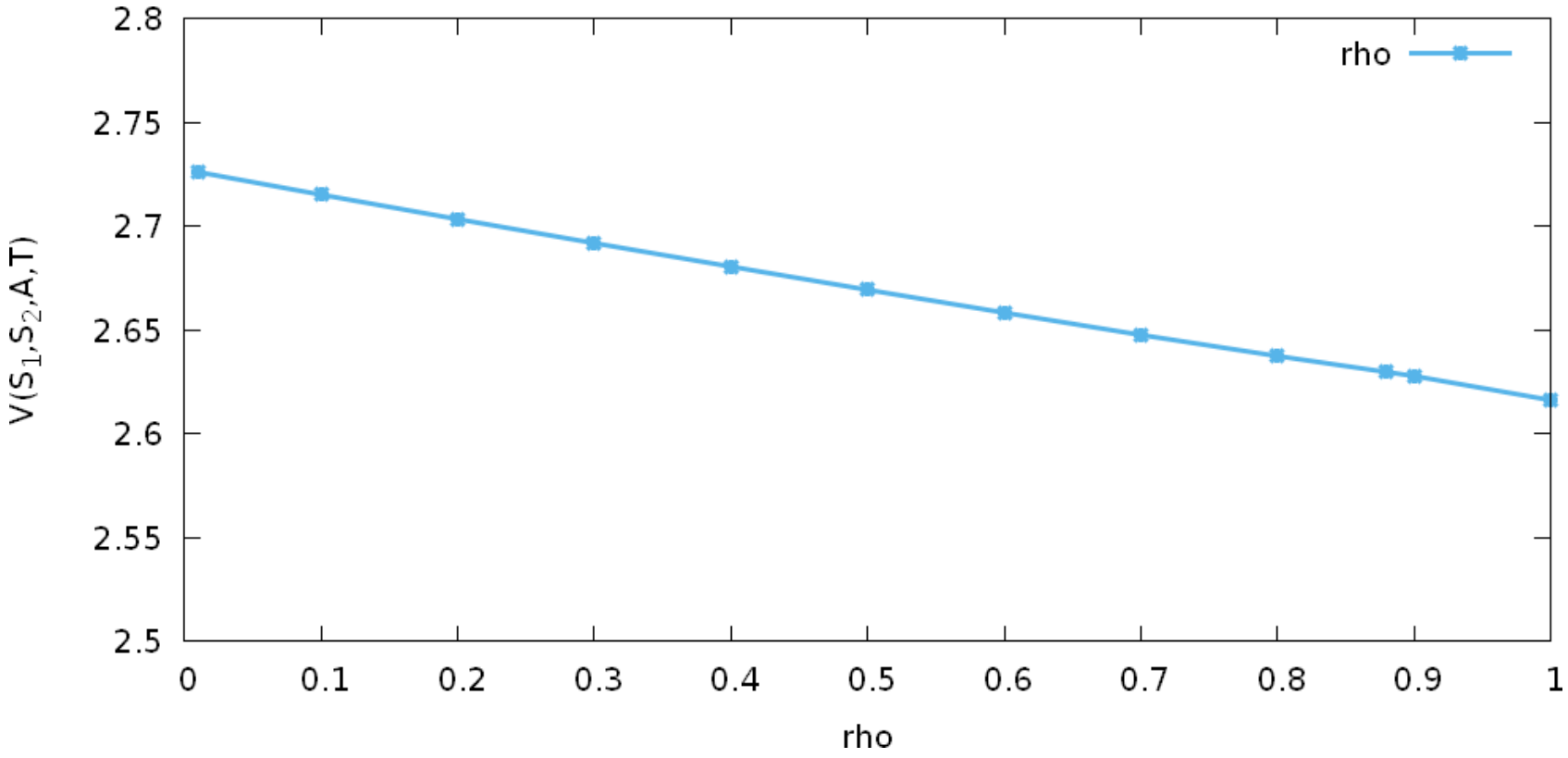
- isolvalues of solutions



- $\#\mathcal{T}_h \approx 4\,000$, results at the reference node $[0.6690, 1.7094]$
  2.62975 (Asian basket) 3.56022 (basket)

Asian basket with fixed strike (3)

- dependency among particular risk sources,
- basket option value at ref. node vs. correlation plot



Conclusion

Achieved results

- fundamentals of option pricing,
- market models based on Black-Scholes equation (PDE),
- combination of Asian and basket options,
- 2 approaches of dimensional reduction w.r.t. payoffs
- nonstationary linear convection-diffusion-reaction problem,
- DG space semi-discretization with backward Euler,
- promising numerical results

Future work

- to add other realistic experiments with real reference values,
- to investigate the sensitivity measures (Delta, Gamma, etc.),
- to extend the approach to other path-dependent options.

References

[1] Y. Achdou and O. Pironneau. *Computational Methods for Option Pricing*. Frontiers in Applied Mathematics. Society for Industrial and Applied Mathematics, Philadelphia, 2005.

[2] F. Hecht. New development in FreeFem++, *Journal of Numerical Mathematics* **20**, no. 3-4, pp. 251–265, 2012.

[3] J.E. Ingersoll, Jr. *Theory of Financial Decision Making*. Rowman & Littlefield Publishers, Inc., New Jersey, 1987

[4] L. Rogers and Z. Shi The Value of an Asian Option, *Journal of Applied Probability* **32**(4), pp. 1077–1088, 1995.

[5] R. Seydel. *Tools for Computational Finance:* 4th *edition*. Springer, 2008.

Thank you for your attention