DG method for the numerical pricing of path-dependent multi-asset options

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Outline

1 Motivation
   - Financial background
   - Basket options
   - Asian options

2 Path-dependent multi-asset options

3 Asian two-asset option with floating strike
   - PDE formulation
   - Weak formulation
   - DG discretization
   - Numerical experiments

4 Asian two-asset option with fixed strike
   - Similar approach
   - Numerical experiments
1 Motivation

2 Path-dependent multi-asset options

3 Asian two-asset option with floating strike

4 Asian two-asset option with fixed strike
Objects of our interest

- to develop efficient, accurate and robust numerical scheme for the simulation of option pricing problem within DG technique,
- to optimize particular steps of the approximation to obtain correct results attainable in a reasonable time,
- to be robust with respect to various types of options as well as market conditions,
- to analyze the impact of various approaches
- to examine the computational error and demandingness
Basic mechanism of options

- **option** - a special case of financial derivatives, contract between two parties about trading the asset at a certain future time

  - **writer** sells **premium** (market price) → **holder** purchases

- **option** has a limited life time: maturity or expiration time \( T \)

- we distinguish:
  - **call** - gives the holder the right to *buy* the underlying for an agreed price \( K \) (strike) by the date \( T \)
  - **put** - gives the holder the right to *sell* the underlying for an agreed price \( K \) by the date \( T \)

What is the price of such a contract?
at expiration $T$ the value of option $V(S, t)$ is clearly given by the payoff function

$$V(S, 0) = \begin{cases} 
\max(S - K, 0), & \text{for a call} \\
\max(K - S, 0), & \text{for a put}
\end{cases}$$
Option value

- at expiration $T$ the value of option $V(S, t)$ is clearly given by the payoff function

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\max(S - K, 0), & \text{for a call} \\
\max(K - S, 0), & \text{for a put}
\end{cases}$$
at expiration $T$ the value of option $V(S, t)$ is clearly given by the payoff function

$$V(S, 0) = \begin{cases} \max(S - K, 0), & \text{for a call} \\ \max(K - S, 0), & \text{for a put} \end{cases}$$

But what about now?
How much would you pay for such an option?
How to calculate a fair value of the option?
at expiration $T$ the value of option $V(S, t)$ is clearly given by the payoff function

$$V(S, 0) = \begin{cases} \max(S - K, 0), & \text{for a call} \\ \max(K - S, 0), & \text{for a put} \end{cases}$$

But what about now?
How much would you pay for such an option?
How to calculate a fair value of the option?
Introduction to option pricing

- a financial asset with tendency $\mu$ and volatility $\sigma$

\[ dS_t = S_t(\mu dt + \sigma dW_t), S_0 \text{ known} \] (1)

- $\mu$ is drift of $S_t$, $\mu = r$, where $r$ is risk-free interest rate
- $W_t$ is a standard Brownian motion (BM)
- for constant $\sigma$ and $r$ we have solution of SDE (1) as

\[ S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} \text{ (geometric BM)} \]

- option value at $t$ is the expected profit at $T$ discounted to $t$:

\[ C_t = e^{-r(T-t)} \mathbb{E}(\max(S_T - K, 0)) \text{ (call)} \]
\[ P_t = e^{-r(T-t)} \mathbb{E}(\max(K - S_T, 0)) \text{ (put)} \]

- due to the put call parity we can either compute calls or puts

\[ C_t - P_t = S_t - Ke^{-r(T-t)} \]
Numerical techniques for option pricing

1. Monte-Carlo simulations

\[ S_{k+1}^n - S_k^n = S_k^n(\mu \delta t + \sigma \sqrt{\delta t} \mathcal{N}(0, 1)), \quad S_0^n = S_0 \]

\[ C_t = e^{-r(T-t)} \frac{1}{N} \sum_{n=1}^{N} (\max(S_T^n - K, 0)) \]

where \( \mathcal{N}(0, 1) \) is approximated by random function

2. Tree methods (Cox, Ross and Rubinstein binomial model)

3. PDE approach within Ito calculus

\[ df(t, S_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS_t dS_t) \]

\[ \rightarrow \text{Black-Scholes (BS) equation} \]
Monte-Carlo vs. PDE approach

Monte-Carlo simulation

- near to the modelisation,
- easy to parallelize,
- sensitivity parameters (Greeks) using Malliavin calculus,
- calibration is very hard

PDE approach

- nearer to the analytical formulas,
- can be parallelized,
- sensitivity parameters (Greeks) are easy to evaluate,
- calibration is feasible
Derivation of BS equation

We make the following assumptions:

- the assets prices follow the lognormal random walk,
- the risk-free interest rate $r$, the volatility $\sigma$ are constant,
- no transactions costs associated with hedging a portfolio,
- no arbitrage possibilities,
- the underlying assets pay no dividends,
- trading of the underlying assets can take place continuously,
- short selling is permitted and the assets are divisible.

We construct a portfolio of option and $\Delta$ shares of the underlying

$$\Pi = V(S, t) - \Delta S$$

where $V$ represents a long position (it is owned) and $\Delta S$ a short position (it is owed)
Derivation of BS equation

(A) the change in our portfolio from $t$ to $t + dt$ is completely riskless (riskless portfolio), described by ODE,

$$d\Pi = r\Pi dt$$

(B) equivalently, the change in our portfolio is also equal to

$$d\Pi = dV - \Delta dS$$

and using Ito lemma with $(dSdS) \approx \sigma^2 S^2 dt$ we have

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt$$

i.e., the portfolio changes by

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \left( \frac{\partial V}{\partial S} - \Delta \right) dS$$
Derivation of BS equation

- Randomness is eliminated when $\Delta = \frac{\partial V}{\partial S}$ in our portfolio (delta hedging, dynamic hedging strategy), i.e.

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

- Putting (A) and (B) together we obtain

$$r\Pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

Finally, from $\Pi = V - \Delta S = V - \frac{\partial V}{\partial S} S$ we get the famous Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$
Black-Scholes (BS) equation

- derivation of BS equation [Black, Scholes, 1973],
- a Nobel prize was awarded in 1997

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]

- \( V = V(S, t; T, K, r, \sigma) \) - value of option
- \( t, \ 0 \leq t \leq T \) - current time, \( T - t \) - time time to expiration
- \( S \) - price of stock at time \( t \), i.e. \( S = S(t), \ S \in (0; +\infty) \)
- \( r, \ r > 0 \) - risk-free interest rate
- \( \sigma, \ \sigma > 0 \) - volatility of price \( S \)
- \( K \) - strike price

- backward linear parabolic differential equation
by transforming the Black-Scholes PDE into the heat equation with reversal time

Black-Scholes formula

\[ C_t = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \]
\[ P_t = -S_t \Phi(-d_1) + K e^{-r(T-t)} \Phi(-d_2) \]

where

\[ d_1 = \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}, \quad d_2 = d_1 - \sigma \sqrt{T-t} \]

and \( \Phi \) denotes the cumulative distribution function of the standard normal distribution.
BS equation – numerical experiments

- real data of the DAX on 15SEPT2011 with implied volatilities,
- expiration date $T = 0.50137$ (half year),
- risk-free interest rate $r = 0.0176$,
- uniform space-time grid with constant mesh step $h$ and constant time step $\tau$,
- DG piecewise $P_3$ (cubic) approximations,
- sensitivity measures (Greeks):
  \[
  \Delta = \frac{\partial u}{\partial x} \quad \text{and} \quad \Gamma = \frac{\partial^2 u}{\partial x^2}
  \]
- we investigate the behaviour of approximation values of option and Greeks w.r.t. the choice of b.c.
- reference DAX data at underlying’s price $x = 5508.238$ (star-symbol in graphs)
Put options - comparison at expiration date

- opt. val. and Greeks (Dirichlet b.c., $K = 4000$), ref. val. (*)

![Graphs showing option value, delta, and gamma versus stock price](image-url)
Put options - comparison at expiration date

- opt. val. and Greeks (Neumann b.c., $K = 4000$), ref. val. (*)
Put options - comparison at expiration date

- opt. val. and Greeks (transparent b.c., $K = 4000$), ref. val.$(\ast)$

[Graphs showing European put options at $T=0.50137$ ($r=0.0176$, $\sigma=0.4594$) with transparent BC, Delta, and Gamma values at different stock prices.]
Limitations and generalizations of the BS model

- idealized assumptions are rarely valid in the real world
  - normal distribution of log-returns
  - constant model parameters
- important market features are not captured by BSM
  - volatility clustering and leverage effect
  - volatility smile
- generalizations of BS framework
  - assumptions of non-constant deterministic (e.g. local volatility) or stochastic parameters (e.g. Heston model)
  - other models of the price movements instead of diffusion one (in BS model) such as jump-diffusion model (leading to PIDE) or pure jump model (e.g. VG model)
Classification of options

- from the holder’s point of view
  - calls - a right to buy
  - puts - a right to sell

- according to the option style
  - European - may only be exercised on expiry
  - American - may be exercised on any trading day on or before expiration

- according to the relation between asset price $S$ and strike $K$
  - out of the money (OTM): $S < K$ (call) and $S > K$ (put)
  - at the money (ATM): $S = K$ (call and put)
  - in the money (ITM): $S > K$ (call) and $S < K$ (put)

- standard (vanilla) option vs. non-standard (exotic) options
  - discontinuous payoff function (e.g. barrier)
  - depend on a basket of several underlying assets (e.g. basket)
  - path-dependent options (e.g. Asian)
Basket options

- generally, dimension $d \geq 2 \Rightarrow$ multi-asset options
- option depending on a basket of two underlying assets

$\{\text{price of } 1^{st} \text{ asset } S_1, \text{ price of } 2^{nd} \text{ asset } S_2\} \Rightarrow \text{value of basket option } V(S_1, S_2, t),$

- movement of stock prices is driven by SDEs

\[
\begin{align*}
    dS_1(t) &= rS_1(t)dt + \sigma_1 S_1(t)dW_1(t) \\
    dS_2(t) &= rS_2(t)dt + \sigma_2 S_2(t)dW_2(t)
\end{align*}
\]

with correlated Wiener processes, i.e., $dW_1(t)dW_1(t) = \rho dt,$

- $S_k(t)$ - prices of stock at time $t$, $0 \leq S_k < \infty$, $k = 1, 2,$
- $r$, $r > 0$ - risk-free interest rate,
- $\sigma_k$, $\sigma_k > 0$ - volatility of the $k$-th price $S_k$, $k = 1, 2$
Motivation
Path-dependent multi-asset options
Asian two-asset option with floating strike
Asian two-asset option with fixed strike

Financial background
Basket options
Asian options

2D Black-Scholes equation

- analogous approach to 1D BS model with risk-free portfolio

\[ \Pi = V(S_1, S_2, t) - \Delta_1 S_1 - \Delta_2 S_2 \]

- application of multidimensional Ito calculus
- delta hedging with \( \Delta_1 = \frac{\partial V}{\partial S_1} \) and \( \Delta_2 = \frac{\partial V}{\partial S_2} \)
- generalization of BS equation into 2D

\[
- \frac{\partial V}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + r S_1 \frac{\partial V}{\partial S_1} + r S_2 \frac{\partial V}{\partial S_2} = r V
\]

\[ V = V(S_1, S_2, t; T, K, r, \sigma_1, \sigma_2, \rho) \] - value of option

\[ t, 0 \leq t \leq T \] - time to expiration, \( T - t \) - current time

- reversal time \( \Rightarrow \) initial condition is given by payoff function

\[
V(S_1, S_2, 0) = \max(\alpha_1 S_1 + \alpha_2 S_2 - K, 0) \quad \text{(call)}
\]

\[
V(S_1, S_2, 0) = \max(K - \alpha_1 S_1 - \alpha_2 S_2, 0) \quad \text{(put)}
\]

\[ \alpha_k > 0, \alpha_1 + \alpha_2 = 1 \]
Basket options – numerical experiments

Parameter settings:

- basket put option with 60% Allianz ($\alpha_1 = 0.6$) and 40% Deutsche Bank ($\alpha_2 = 0.4$), strike at 40 Euro
- current date: September 13, 2011 (Tuesday),
- piecewise constant volatilities $\sigma_1$ and $\sigma_2$,
- expiration date $T = 0.257534$ (i.e. 94 days),
- risk-free interest rate $r = 0.01557$ p.a.,
- Pearson linear correlation $\rho = 0.88$
- maximal stock prices $S_{1}^{\text{max}} = 130.0$ and $S_{2}^{\text{max}} = 220.0$,
- two stocks trades at $S_{1}^{\text{ref}} = 59.79$ and $S_{2}^{\text{ref}} = 23.40$

Implementation settings:

- adaptively refined grid, constant time step $\tau = 1/365$
- piecewise $P_1$ (linear) approximations, sparse solver GMRES
Basket options - numerical results

- space-time plot of solution at maturity
Asian options

- be introduced to avoid the manipulation of prices on expiration date → less sensitive to market fluctuations,
- a representative of the class of path dependent options,
- **value of Asian option** \( V = V(S, A, t) \) depends also on the average \( A \) of the price of the underlying assets
- we distinguish

  - fixed strike
  - floating strike
  - continuous
  - discrete
  - arithmetic averaging
  - geometric averaging

- average of the stock price at time \( t \) is given by

  \[
  A = \frac{1}{t} \int_{0}^{t} S(u)du
  \]
Asian options

- Movement of the stock price is driven by SDE
  \[ dS(t) = rS(t)dt + \sigma S(t)dW(t) \]
- Movement of the average-variable is driven by ODE
  \[ dA(t) = \frac{1}{t} \left( S(t) - A(t) \right) dt \]
- \( r, r > 0 \) - risk-free interest rate (constant),
- \( \sigma, \sigma > 0 \) - volatility of the price \( S \) (constant)
- Analogous approach to 1D BS model with risk-free portfolio
  \[ \Pi = V(S, A, t) - \Delta S \]
- Application of multidimensional Ito calculus
- Delta hedging with \( \Delta = \frac{\partial V}{\partial S} \)
- We obtain 2D PDE model of Asian options
2D PDE model of Asian options

- continuous arithmetic Asian options with fixed strike

\[
\frac{\partial V}{\partial t} - \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - rS \frac{\partial V}{\partial S} - \frac{S - A}{T - t} \frac{\partial V}{\partial A} + rV = 0
\]

\[
V = V(S, A, t; T, K, r, \sigma) - \text{value of option}
\]

- parabolic PDE degenerated in variable \( A \)
- reversal time \( \Rightarrow \) initial condition is given by payoff function,

\[
V(S, A, 0) = \max(A - K, 0) \quad \text{(call with fixed strike)}
\]

\[
V(S, A, 0) = \max(K - A, 0) \quad \text{(put with fixed strike)}
\]
Asian options – numerical experiments

Parameter settings:
- Asian puts on German stock market index DAX,
- real data of the DAX on 15SEPT2011 with implied volatilities,
- expiration date $T = 0.50137$ (half year, 183 days),
- strike $K = 4000$,
- constant volatility $\sigma = 0.4594$,
- constant risk-free interest rate $r = 0.0176$,
- maximal stock price $S_{\text{max}} = A_{\text{max}} = 2K$ (usually),
- reference node $S_{\text{ref}} = 5508$

Implementation settings:
- uniform grid, constant time step $\tau = 1/365$,
- piecewise $P_1$ (linear) approximations, sparse solver GMRES
Asian options - numerical results (1)

- space-time plot of solution at $t = 0$ days (initial state)
Asian options - numerical results (1)

- space-time plot of solution at $t = 92$ days (mid state)
Asian options - numerical results (1)

- space-time plot of solution at $t = 183$ days (final state)
Asian options - numerical results (2)

- Isovalues of solution at $t = 0$ days (initial state)
Asian options - numerical results (2)

- isovalues of solution at $t = 92$ days (mid state)
Asian options - numerical results (2)

- isovales of solution at $t = 183$ days (final state)
Motivation

Path-dependent multi-asset options
Asian two-asset option with floating strike
Asian two-asset option with fixed strike
General model of path-dependent multi-asset options

- combination of basket and path-dependent options
- simplification: basket options with 2 assets
- price function $V = V(S_1(t), S_2(t), A(t), t)$ depends on two correlated price processes
  \[ dS_k(t) = rS_k(t)dt + \sigma_k S_k(t)dW_k(t), \quad k = 1, 2 \]
- and on path-dependent variable (in general form)
  \[ A(t) = \frac{1}{t} \int_0^t g(S_1(u), S_2(u), u)du \] (continuous case)
- driven by ODE (with drift component only)
  \[ dA(t) = \frac{1}{t} \left( g(S_1(t), S_2(t), t) - A(t) \right) dt \]
- where $g$ is given according to specific type of option
General model of path-dependent multi-asset options

- analogous approach to basket and Asian options with portfolio
- application of multidimensional Ito calculus
- delta hedging with $\Delta_1 = \frac{\partial V}{\partial S_1}$ and $\Delta_2 = \frac{\partial V}{\partial S_2}$
- 3D PDE model of path-dependent two-asset options

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + r S_1 \frac{\partial V}{\partial S_1} + r S_2 \frac{\partial V}{\partial S_2} + \frac{1}{t} \left( g(S_1, S_2, t) - A \right) \frac{\partial V}{\partial A} - r V = 0
\]

$V = V(S_1, S_2, A, t; T, K, r, \sigma_1, \sigma_2, \rho)$ - value of option

t, $0 \leq t \leq T$ - current time, $T - t$ - time to expiration

- parabolic PDE degenerated in variable $A$
- backward PDE $\Rightarrow$ terminal condition is given by payoff
Asian two-asset basket option

- various forms of $g \rightarrow$ different path-dependent options
- Asian option = average options (subclass of path-dep. opt.)
  - continuous arithmetic averaging
    $$A(t) = \frac{1}{t} \int_0^t \left( \alpha_1 S_1(u) + \alpha_2 S_2(u) \right) du$$
  - continuous geometric averaging (not considered here)
    $$A(t) = \frac{1}{t} \exp \left( \int_0^t \left( \alpha_1 S_1(u) + \alpha_2 S_2(u) \right) du \right)$$
  - continuous weighted arithmetic averaging
    $$A(t) = \frac{1}{t} \int_0^t a(t - u) \left( \alpha_1 S_1(u) + \alpha_2 S_2(u) \right) du$$

where $\alpha_1, \alpha_2 > 0$ are weights with $\alpha_1 + \alpha_2 = 1$
and $a(\cdot) \geq 0$ s.t. $\int_0^\infty a(\xi) d\xi < \infty$
Asian two-asset basket option: payoff function

- function of average over some time period prior to expiry
- 2 ways how the average is incorporated into payoff
  - average strike option (= option with floating strike)
    \[
    V(S, A, T) = \max(\alpha_1 S_1(T) + \alpha_2 S_2(T) - A(T), 0) \quad \text{(call)}
    \]
    \[
    V(S, A, T) = \max(A(T) - \alpha_1 S_1(T) - \alpha_2 S_2(T), 0) \quad \text{(put)}
    \]
  - average rate option (= option with fixed strike)
    \[
    V(S, A, T) = \max(A(T) - K, 0) \quad \text{(call)}
    \]
    \[
    V(S, A, T) = \max(K - A(T), 0) \quad \text{(put)}
    \]
- we focus on

Asian two-asset basket option with floating strike
Asian two-asset basket option with fixed strike
1. Motivation

2. Path-dependent multi-asset options

3. Asian two-asset option with floating strike

4. Asian two-asset option with fixed strike
Motivation
Path-dependent multi-asset options
Asian two-asset option with floating strike
Asian two-asset option with fixed strike

PDE formulation
Weak formulation
DG discretization
Numerical experiments

3D PDE model

- continuous arithmetic Asian two-asset basket option with floating strike

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + r S_1 \frac{\partial V}{\partial S_1} + r S_2 \frac{\partial V}{\partial S_2} + \frac{1}{t} \left( \alpha_1 S_1 + \alpha_2 S_2 - A \right) \frac{\partial V}{\partial A} - rV &= 0
\end{align*}
\]

\[
V = V(S_1, S_2, A, t; T, K, r, \sigma_1, \sigma_2, \rho) - \text{value of option}
\]

- \( t, \ 0 \leq t \leq T \) - current time, \( T - t \) - time to expiration
- \( S_1, S_2, A > 0 \) - underlying assets and average

- backward PDE \( \Rightarrow \) terminal condition is given by payoff,
- for \( t = 0 \) we obtain 2D Black-Scholes model
3D PDE model - boundary conditions (1)

- localization problem on bounded domain
- at far-field boundary (from asymptotic behaviour at infinity)

\[ \text{if } V \text{ is call} \left\{ \begin{array}{l}
\lim_{S_1 \to \infty} \frac{\partial V}{\partial S_1}(S_1, S_2, A, t) = \alpha_1, \\
\lim_{S_2 \to \infty} \frac{\partial V}{\partial S_2}(S_1, S_2, A, t) = \alpha_2, \\
\lim_{A \to \infty} V(S_1, S_2, A, t) = 0,
\end{array} \right. \]

\[ \text{and} \]

\[ \text{if } V \text{ is put} \left\{ \begin{array}{l}
\lim_{S_1 \to \infty} V(S_1, S_2, A, t) = 0, \\
\lim_{S_2 \to \infty} V(S_1, S_2, A, t) = 0, \\
\lim_{A \to \infty} \frac{\partial V}{\partial A}(S_1, S_2, A, t) = 1,
\end{array} \right. \]
on planes $S_1 = 0$ and $S_2 = 0$ (concept of extrapolated b.c.)

if $V$ is call

\[
\begin{align*}
V(0, S_2, A, t) &:= \lim_{\varepsilon \to 0^+} V(\varepsilon, S_2, A, t), \\
V(S_1, 0, A, t) &:= \lim_{\varepsilon \to 0^+} V(S_1, \varepsilon, A, t)
\end{align*}
\]

and

if $V$ is put

\[
\begin{align*}
\frac{\partial V}{\partial S_1}(0, S_2, A, t) &:= \lim_{\varepsilon \to 0^+} \frac{\partial V}{\partial S_1}(\varepsilon, S_2, A, t), \\
\frac{\partial V}{\partial S_2}(S_1, 0, A, t) &:= \lim_{\varepsilon \to 0^+} \frac{\partial V}{\partial S_2}(S_1, \varepsilon, A, t)
\end{align*}
\]

only one boundary condition in $A$-direction is prescribed
Dimensional reduction

- to avoid the higher complexity of 3D model
- change of variables (approach from [Ingersoll])

\[ x_1 = \frac{S_1}{A}, \quad x_2 = \frac{S_2}{A}, \quad \hat{t} = T - t \]

- transformation of price function

\[ u(x, \hat{t}) = \frac{1}{A} V(S_1, S_2, A, t), \quad \text{where} \ x = [x_1, x_2] \]

- transformation of payoffs (= initial conditions)

\[ u(x, 0) = \frac{1}{A} V(S_1, S_2, A, T) = \max(\alpha_1 x_1 + \alpha_2 x_2 - 1, 0) \quad \text{(call)} \]
\[ u(x, 0) = \frac{1}{A} V(S_1, S_2, A, T) = \max(1 - \alpha_1 x_1 - \alpha_2 x_2, 0) \quad \text{(put)} \]

- transformation of PDE: homogeneity w.r.t. \( A \)

\[ \longrightarrow \text{separation of variable} \ A \longrightarrow \text{dimensional reduction from 3D to 2D} \]
2D reformulation

- localization on a truncated computational domain:
  \[ [x_1, x_2] \in \mathbb{R}_0^2 \longrightarrow [x_1, x_2] \in \Omega = (0, x_{1\text{max}}) \times (0, x_{2\text{max}}) \]

- convection-diffusion-reaction equation in divergence-free form

We seek \( u : Q_T = \Omega \times (0, T) \rightarrow \mathbb{R} \) such that

\[
\frac{\partial u}{\partial \hat{t}} - \text{div } (D(x) \cdot \nabla u) + \nabla \cdot \vec{f}(x, \hat{t}, u) + \gamma(x, \hat{t})u = 0 \quad \text{in } Q_T, \ (2)
\]

\[ B(u) = 0 \quad \text{on } \partial \Omega, \]

\[ u(x, 0) = u^0(x), \quad x \in \Omega, \]

where

\[
D(x) \equiv \{ D(x)_{kl} \}_{k,l=1}^2 = \frac{1}{2} \begin{pmatrix} \sigma_1^2 x_1^2 & \rho \sigma_1 \sigma_2 x_1 x_2 \\ \rho \sigma_1 \sigma_2 x_1 x_2 & \sigma_2^2 x_2^2 \end{pmatrix},
\]
2D reformulation

- localization on a truncated computational domain:
  \[[x_1, x_2] \in \mathbb{R}^2 \rightarrow [x_1, x_2] \in \Omega = (0, x_1^{max}) \times (0, x_2^{max})\]

- convection-diffusion-reaction equation in divergence-free form

We seek \( u : Q_T = \Omega \times (0, T) \rightarrow \mathbb{R} \) such that

\[
\begin{align*}
\frac{\partial u}{\partial \hat{t}} - \text{div} (D(x) \cdot \nabla u) + \nabla \cdot \vec{f}(x, \hat{t}, u) + \gamma(x, \hat{t})u &= 0 \quad \text{in } Q_T, (2) \\
B(u) &= 0 \quad \text{on } \partial \Omega, \\
u(x, 0) &= u^0(x), \quad x \in \Omega,
\end{align*}
\]

where \( \vec{f}(x, \hat{t}, u) = (f_1(x, \hat{t}, u), f_2(x, \hat{t}, u))^T \) with

\[
f_k(x, \hat{t}, u) = \left( \sigma_k^2 + \frac{1}{2} \rho \sigma_1 \sigma_2 - r + \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T - \hat{t}} \right) x_k \cdot u, \quad k = 1, 2
\]
2D reformulation

- Localization on a truncated computational domain:

\[
[x_1, x_2] \in \mathbb{R}^2_0 \rightarrow [x_1, x_2] \in \Omega = (0, x_{1 \text{max}}) \times (0, x_{2 \text{max}})
\]

- Convection-diffusion-reaction equation in divergence-free form

We seek \( u : Q_T = \Omega \times (0, T) \rightarrow \mathbb{R} \) such that

\[
\frac{\partial u}{\partial \hat{t}} - \text{div} \left( D(x) \cdot \nabla u \right) + \nabla \cdot \tilde{f}(x, \hat{t}, u) + \gamma(x, \hat{t})u = 0 \quad \text{in } Q_T, (2)
\]

\[
B(u) = 0 \quad \text{on } \partial \Omega,
\]

\[
u(x, 0) = u^0(x), \quad x \in \Omega,
\]

where

\[
\gamma(x, \hat{t}) = 3r - \frac{4\alpha_1 x_1 + 4\alpha_2 x_2 - 3}{T - \hat{t}} - \sigma_1^2 - \sigma_2^2 - \rho \sigma_1 \sigma_2
\]
2D reformulation

- localization on a truncated computational domain:
  \[ [x_1, x_2] \in \mathbb{R}^2 \rightarrow [x_1, x_2] \in \Omega = (0, x_1^{\text{max}}) \times (0, x_2^{\text{max}}) \]

- convection-diffusion-reaction equation in divergence-free form

Let \( u : Q_T = \Omega \times (0, T) \rightarrow \mathbb{R} \) such that

\[
\frac{\partial u}{\partial \hat{t}} - \text{div} (D(x) \cdot \nabla u) + \nabla \cdot \mathbf{f}(x, \hat{t}, u) + \gamma(x, \hat{t})u = 0 \quad \text{in } Q_T, \tag{2}
\]

- boundary conditions \( B(u) \) specified later

\( \hat{t} \) - time to maturity \( \rightarrow \) i.c. given by payoffs \( u^0 \)
Let $\partial \Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. W.l.o.g. we consider the case of the put option.
setting boundary conditions in consistency with the vector field induced by physical fluxes, i.e. \( \left( \frac{\partial f_1}{\partial u}, \frac{\partial f_2}{\partial u} \right) \)
Boundary conditions (3)

- reformulation of original b.c.
- homogeneous Neumann b.c. on axes $x_1 = 0$ and $x_2 = 0$

\[
(D(x) \cdot \nabla u(x_1, x_2, \hat{t})) \cdot \vec{n} = 0, \quad \text{on } \Gamma_i, i = 1, 3, \hat{t} > 0,
\]

where $\vec{n}$ is the unit outer normal vector,

- asymptotic behaviour of options at infinity

\[
u(x_1, x_2, \hat{t}) = 0 \quad \text{on } \Gamma_2, \hat{t} \in (0, T).
\]

- b.c. for calls follow from put-call parity
Weighted Lebesgue space

- standard Lebesgue space

$$L^2(\Omega) = \left\{ v \in L^1_{loc}(\Omega) : \int_{\Omega} v^2(x)dx < +\infty \right\}$$

with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$

$$\langle u, v \rangle = \int_{\Omega} u(x) \cdot v(x)dx, \quad \| u \| = \sqrt{(u, u)}$$

- $w$ is weight, i.e. $w : \mathbb{R}^2 \rightarrow [0, \infty)$, $w \in L^1_{loc}(\Omega)$ and $w \neq 0$,

- weighted Lebesgue space

$$L^2_w = L^2(\Omega, w) = \left\{ v \in L^1_{loc}(\Omega) : \int_{\Omega} \sqrt{w} \cdot v \in L^2(\Omega) \right\}$$

with scalar product $\langle \cdot, \cdot \rangle_w$ and induced norm $\| \cdot \|_w$

$$\langle u, v \rangle_w = \int_{\Omega} u(x) \cdot v(x) \cdot w(x)dx, \quad \| u \|_w = \| \sqrt{w} \cdot u \|.$$
Weighted Sobolev space

- weighted Sobolev space

\[ H^1_w = H^1(\Omega, w_1, w_2) = \left\{ v \in L^2(\Omega) : \sqrt{w_i} \cdot \frac{\partial v}{\partial x_i} \in L^2(\Omega), \ i = 1, 2 \right\} \]

with the norm

\[ \| u \|_{1,w} = \left( \| u \|^2 + \left\| \frac{\partial v}{\partial x_1} \right\|_{w_1}^2 + \left\| \frac{\partial v}{\partial x_2} \right\|_{w_2}^2 \right)^{1/2} \]

- in our case, we take: \( w_1(x) = x_1^2 \) and \( w_2(x) = x_2^2 \),

- to treat with b.c., we introduce space:

\[ H_{w,0}^1 = \left\{ v \in H^1_w : v|_{\Gamma_D} = 0 \right\} \]
Properties of $H^1_w$

The space $H^1_w$ has the following properties:

(a) $H^1_w$ is separable,

(b) $\mathcal{D}(\overline{\Omega}) = \{ v \in C_0^\infty(\mathbb{R}^2) : v|_\Omega \}$ is dense in $H^1_w$,

(c) $H^1_w$ is dense in $L^2(\Omega)$ with continuous embedding $H^1_w \subseteq L^2(\Omega)$,

(d) the seminorm

$$|u|_{1,w} = \left( \sum_{i=1}^2 \left\| \frac{\partial v}{\partial x_i} \right\|_{w_i}^2 \right)^{1/2}$$

is in fact a norm on $H^1_{w,0}$ equivalent to $\| \cdot \|_{1,w}$. 
Variational formulation (1)

- we define linear partial differential operator

\[
\mathcal{L}_t u = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i} (D(x) \cdot \nabla u) + \sum_{i=1}^{2} a_i(x, \hat{t}) \frac{\partial u}{\partial x_i} + \left( r - \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T - \hat{t}} \right) u.
\]

where

\[
a_1(x, \hat{t}) = \left( \sigma_1^2 + \frac{1}{2} \rho \sigma_1 \sigma_2 - r + \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T - \hat{t}} \right) x_1,
\]

\[
a_2(x, \hat{t}) = \left( \sigma_2^2 + \frac{1}{2} \rho \sigma_1 \sigma_2 - r + \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T - \hat{t}} \right) x_2.
\]
standard approach leads to

$$\left( \frac{\partial u}{\partial \hat{t}}, \nu \right) + (\mathcal{L}_t u, \nu) = 0 \quad \forall \, \nu \in H^1_{w,0}, \text{ a.e. } \hat{t} \in (0, T),$$

where

$$\mathcal{L}_t u, \nu = \int_{\Omega} D(x) \cdot \nabla u \cdot \nabla \nu \, dx + \sum_{i=1}^{2} \int_{\Omega} a_i(x, \hat{t}) \frac{\partial u}{\partial x_i} \, \nu \, dx$$

$$+ \int_{\Omega} \left( r - \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T - \hat{t}} \right) uv \, dx.$$
Weak solution

- assume that there exists a function

\[ u_g \in C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_w), \quad \frac{\partial u_g}{\partial \hat{t}} \in L^2(0, T; H'_w) \]

with \( u_g \rvert_{\Gamma_D} = g \) (prescribed Dirichlet b.c.)

Find \( u: u - u_g \in L^2(0, T; H^1_{w,0}), \ u \in C^0([0, T]; L^2(\Omega)), \) s.t.
\[ \frac{\partial u}{\partial \hat{t}} \in L^2(0, T; H'_w) \] satisfying \( u \bigg|_{\hat{t}=0} = u^0 \) in \( \Omega \) \( (u^0 \in L^2(\Omega)) \) and

\[ \left( \frac{\partial u}{\partial \hat{t}}, \nu \right) + (\mathcal{L}_t u, \nu) = 0 \quad \forall \nu \in H^1_{w,0}, \text{ a.e. } \hat{t} \in (0, T), \quad (3) \]
Properties of $\mathcal{L}_t$

- **boundedness:** $\exists C_B = C_B(\hat{t}) > 0$ s.t.
  \[
  (\mathcal{L}_t u, v) \leq C_B(\hat{t})|u|_{1,w} |v|_{1,w}, \quad \forall u, v \in H^1_{w,0}, \hat{t} \in (0, T^*)
  \]

- **Gårding inequality:** $\exists C_G = C_G(\hat{t}), C_g = C_g(\hat{t}) > 0$ s.t.
  \[
  (\mathcal{L}_t u, u) \geq C_G(\hat{t})|u|_{1,w}^2 - c_g(\hat{t})\|u\|^2, \quad \forall u \in H^1_{w,0}, \hat{t} \in (0, T^*)
  \]

- **Gårding inequality $\Rightarrow$ ellipticity (strict positivity) via easy transformation**
  \[
  u = e^{\lambda \hat{t}} \cdot z, \quad \lambda = \max_{\hat{t} \in [0, T^*]} c_g(\hat{t}), \quad z \in H^1_{w}, \text{ i.e.}
  \]
  \[
  \left(\frac{\partial z}{\partial \hat{t}}, v\right) + (\mathcal{L}_t z, v) + \lambda(z, v) = 0 \quad \forall v \in H^1_{w,0}, \text{ a.e. } \hat{t} \in (0, T^*)
  \]
  \[
  \text{with } z|_{\hat{t}=0} = u^0 \in L^2(\Omega) \text{ and } 0 < T^* < T.
  \]
Unique solvability

Theorem

Problem (3) has a unique weak solution.

Proof (sketch 1/2)

Abstract theory: Let $H$ and $V$ be Hilbert spaces such that $V$ is embedded continuously and densely in $H$. Let $A_t(\cdot, \cdot)$ be continuous bilinear form on $V \times V$. The abstract variational parabolic (AVP) problem associated to the triple $(H, V, A_t(\cdot, \cdot))$ is the following: Given $f(\hat{t}) \in L^2(0, T, V')$ and $u^0 \in H$, find $u \in W(0, T, V, V') = \{u \in L^2(0, T, V) : \frac{du}{d\hat{t}} \in L^2(0, T, V')\}$ s.t.

\[
\frac{\partial}{\partial \hat{t}}(u(\hat{t}), v)_H + A_t(u(\hat{t}), v) = \langle f(\hat{t}), v \rangle \quad \forall v \in V,
\]

$u(0) = u^0$. 

Jiří Hozman

DGM for path-dependent multi-asset options

PANM 18 June '16 50 / 86
Unique solvability

Theorem
Problem (3) has a unique weak solution.

Proof (sketch 2/2)
If the form $A_t(\cdot, \cdot)$ is coercive, then AVP problem admits a unique solution for all $u^0 \in H$.

In our case, we take

\[ T = T^* \]
\[ H = L^2(\Omega), \quad V = H^1_{w,0} \]
\[ \langle f(\hat{t}), v \rangle = -\frac{\partial}{\partial \hat{t}}(u_g(\hat{t}), v)_{H} - A_t(u_g(\hat{t}), v), \]
\[ A_t(u, v) = (\mathcal{L}_t u, v) \]
Discontinuous Galerkin approximations

- exact solution of (2) is difficult to find,
- we construct partitions $\mathcal{T}_h$ of $\Omega$,
- we define space $S_{hp} \approx H^1_w(\Omega)$ over $\mathcal{T}_h$ with $dim(S_{hp}) < \infty$
- we derive DG semi-discrete formulation,
- we discretize in temporal variable,
- we obtain fully discrete DG numerical scheme,
- we construct basis of space $S_{hp}$,
- we assemble a system matrix and right-hand side,
- we solve system of linear algebraic equations,
- we obtain DG solution at each time level.
Triangulations

- let $\mathcal{T}_h$, $h > 0$ be a partition of $\bar{\Omega}$,
- $\mathcal{T}_h = \{K\}_{K \in \mathcal{T}_h}$, $K$ are polygons (nonconvex, hanging nodes),
- $h_K = \text{diam}(K)$, $h = \max_{K \in \mathcal{T}_h} h_K$,
- let $\mathcal{F}_h = \{\Gamma\}_{\Gamma \in \mathcal{F}_h}$ be a set of all edges of $\mathcal{T}_h$,
- we distinguish
  a) inner edges $\mathcal{F}_h^I$
  b) Dirichlet edges $\mathcal{F}_h^D$
  c) Neumann edges $\mathcal{F}_h^N$

$$\Rightarrow \mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^D \cup \mathcal{F}_h^N,$$
Spaces of discontinuous functions

- let $p_K \geq 1, K \in \mathcal{T}_h$ be local polynomial degree,
- we set vector $p \equiv \{p_K, K \in \mathcal{T}_h\}$,
- over $\mathcal{T}_h$ we define:
  - broken weighted Sobolev space
    \[
    H^1_w(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^1_w(K) \ \forall \ K \in \mathcal{T}_h\},
    \]
    where
    \[
    H^1_w(K) = \left\{ v \in L^2(K); x_1 \frac{\partial v}{\partial x_1} \in L^2(K) \land x_2 \frac{\partial v}{\partial x_2} \in L^2(K) \right\},
    \]
  - the space of piecewise polynomial functions
    \[
    S_{hp} \equiv \{v; v \in L^2(\Omega), v|_K \in P_{p_K}(K) \ \forall K \in \mathcal{T}_h\} \subset H^1_w(\Omega, \mathcal{T}_h)
    \]
Example of a function from $S_{hp} \subset H^1_w(\Omega, T_h)$
Notation - trace, mean value, jump

- \( v|_{\Gamma}^{(p)} \equiv \text{trace of } v|_{K_p} \text{ on } \Gamma \) and \( v|_{\Gamma}^{(n)} \equiv \text{trace of } v|_{K_n} \text{ on } \Gamma \),
- \( \langle v\rangle_{\Gamma} = \frac{1}{2} \left( v|_{\Gamma}^{(p)} + v|_{\Gamma}^{(n)} \right) \) and \( [v]_{\Gamma} = v|_{\Gamma}^{(p)} - v|_{\Gamma}^{(n)} \),
- \( \langle v\rangle_{\Gamma} \equiv [v]_{\Gamma} \equiv v|_{\Gamma}^{(p)} , \Gamma \subset \partial \Omega \).
Space semi-discretization

- let $u$ be a strong (regular) solution on $\Omega$,
- we multiply (2) by $v \in H^2_w(\Omega, T_h)$,
- integrate over each $K \in T_h$,
- apply Green’s theorem,
- sum over all $K \in T_h$,
- we include additional terms vanishing for regular solution,
- we obtain the identity

$$
\left( \frac{\partial u}{\partial \hat{t}}(\hat{t}), v \right) + a_h(u(\hat{t}), v) + b_h(u(\hat{t}), v) + J_h(u(\hat{t}), v) \\
+ \gamma(\hat{t})(u(\hat{t}), v) = \ell_h(v)(\hat{t}) \quad \forall v \in H^2_w(\Omega, T_h) \forall \hat{t} \in (0, T)
$$
DG formulation

\[
\left( \frac{\partial u}{\partial t}, v \right) + a_h(u, v) + b_h(u, v) + J_h(u, v) + \gamma(u, v) = \ell_h(v)(\hat{t})
\]

- time derivation, \( u = u(t) \in H^2_w(\Omega, T_h), \hat{t} \in (0, T) \),
- semi-implicit linearization of backward Euler method.
DG formulation

\[
\left( \frac{\partial u}{\partial t}, v \right) + a_h(u, v) + b_h(u, v) + J_h(u, v) + \gamma(u, v) = \ell_h(v)(\hat{t})
\]

- diffusion terms:

\[
a_h(u, v) = \sum_{K \in \mathcal{T}_h} \int_K D(x) \nabla u \cdot \nabla v \, dx
\]

\[
- \sum_{\Gamma \in \mathcal{F}_h^I \cup \mathcal{F}_h^D} \int_{\Gamma} \left\langle D(x) \nabla u \cdot \vec{n} \right\rangle [v] \, dS
\]

\[
+ \sum_{\Gamma \in \mathcal{F}_h^I \cup \mathcal{F}_h^D} \int_{\Gamma} \left\langle D(x) \nabla v \cdot \vec{n} \right\rangle [u] \, dS,
\]

- \( [u]_{\Gamma} = 0, \left\langle D(x) \nabla u \right\rangle_{\Gamma} = D(x) \nabla u|^{(p)}_{\Gamma} = D(x) \nabla u|^{(n)}_{\Gamma}, \forall \Gamma \in \mathcal{F}_h^I \)
DG formulation

\[
\left( \frac{\partial u}{\partial t}, v \right) + a_h(u, v) + b_h(u, v) + J_h(u, v) + \gamma(u, v) = \ell_h(v)(\hat{t})
\]

- **convection terms:**
  \[
  b_h(u, v) = - \sum_{K \in T_h} \int_K \bar{a}(x, \hat{t}) u \cdot \nabla v \, dx + \sum_{\Gamma \in F_h} \int_{\Gamma} H \left( u|_{\Gamma}^{(p)}, u|_{\Gamma}^{(n)}, \bar{n}\Gamma \right) [v] \, dS + \sum_{\Gamma \in F_h \setminus F^l_h} \int_{\Gamma} H \left( u|_{\Gamma}^{(p)}, u^*|_{\Gamma}, \bar{n}\Gamma \right) [v] \, dS
  \]

- **physical flux \Leftrightarrow numerical flux:**
  \[
  \bar{a}(x, \hat{t}) u \cdot \bar{n}|_{\Gamma} \approx H \left( u|_{\Gamma}^{(p)}, u|_{\Gamma}^{(n)}, \bar{n}\Gamma \right), \quad \Gamma \subset K_p \cap K_n.
  \]
DG formulation

\[
\left( \frac{\partial u}{\partial t}, \nu \right) + a_h(u, \nu) + b_h(u, \nu) + J_h(u, \nu) + \gamma(u, \nu) = \ell_h(\nu)(\hat{t})
\]

- numerical flux (concept of upwinding):

\[
H \left( u|_{\Gamma}^{(p)}, u|_{\Gamma}^{(n)}, \bar{n}_{\Gamma} \right) = \begin{cases} 
\bar{a}(x, \hat{t}) u|_{\Gamma}^{(p)} \cdot \bar{n}_{\Gamma}, & \text{if } E > 0 \\
\bar{a}(x, \hat{t}) u|_{\Gamma}^{(n)} \cdot \bar{n}_{\Gamma}, & \text{if } E \leq 0
\end{cases}
\]

where \( E = \sum_{i=1}^{2} a_i(x, \hat{t}) n_i \), \( \bar{a}(x, \hat{t}) = (a_1(x, \hat{t}), a_2(x, \hat{t})) \) with

\[
a_k(x, \hat{t}) = \left( \sigma_k^2 + \frac{1}{2} \rho \sigma_1 \sigma_2 - r + \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T - \hat{t}} \right) x_k, \quad k = 1, 2,
\]

- \( u^*|_{\Gamma} \) is chosen according to boundary conditions.
Motivation
Path-dependent multi-asset options
Asian two-asset option with floating strike
Asian two-asset option with fixed strike

DG formulation

\[
\left( \frac{\partial u}{\partial t}, v \right) + a_h(u, v) + b_h(u, v) + J_h(u, v) + \gamma(u, v) = \ell_h(v)(\hat{t})
\]

- interior and boundary penalty:
  \[
  J_h(u, v) = \sum_{\Gamma \in F_h^I} \int_{\Gamma} \frac{\omega}{|\Gamma|} [u] [v] \, dS + \sum_{\Gamma \in F_h^D} \int_{\Gamma} \frac{\omega}{|\Gamma|} u v \, dS,
  \]
  where
  \- $|\Gamma|$ is the length of edge $\Gamma$,
  \- weighted parameter $\omega = \frac{\sigma_{\min}^2}{4} (1 - |\rho|)$,
  \- $u$ is regular $\Rightarrow [u]_{\Gamma} = 0 \Rightarrow \int_{\Gamma} [u] [v] \, dS = 0$, $\forall \Gamma \in F_h^I$. 

Jiří Hozman
DGM for path-dependent multi-asset options
PANM 18 June '16 57 / 86
DG formulation

\[
\left( \frac{\partial u}{\partial t}, v \right) + a_h(u, v) + b_h(u, v) + J_h(u, v) + \gamma(u, v) = \ell_h(v)(\hat{t})
\]

- reaction terms with factor:
  \[
  \gamma(x, \hat{t}) = 3r - \frac{4\alpha_1 x_1 + 4\alpha_2 x_2 - 3}{T - \hat{t}} - \sigma_1^2 - \sigma_2^2 - \rho \sigma_1 \sigma_2
  \]

- \((\text{broken})\) \(L^2(\Omega)\)-inner product
  \[
  (u, v) = \int_{\Omega} uv \, dx = \sum_{K \in T_h} \int_K uv \, dx
  \]
DG formulation

\[
\left( \frac{\partial u}{\partial t}, v \right) + a_h(u, v) + b_h(u, v) + J_h(u, v) + \gamma(u, v) = \ell_h(v)(\hat{t})
\]

- right-hand-side ⇔ boundary conditions

\[
\ell_h(v)(\hat{t}) = \sum_{\Gamma \in \mathcal{F}_h^D} \int_{\Gamma} D(x) \nabla v \cdot \vec{n}_\Gamma u^*(\hat{t}) dS + \sum_{\Gamma \in \mathcal{F}_h^D} \int_{\Gamma} \frac{\omega}{|\Gamma|} u^*(\hat{t}) v dS
\]

- terms from homogeneous Neumann b.c. vanish.
Semi-discrete DG scheme

- $S_{hp} \subset H^2_w(\Omega, T_h) \Rightarrow (4)$ makes sense for $u_h, v_h \in S_{hp}$
- we define new bilinear form

$$C_h(u, v) := a_h(u, v) + b_h(u, v) + J_h(u, v) + \gamma(u, v)$$

Semi-discrete DG solution

We say that $u_h$ is a semi-discrete DG solution iff

a) $u_h \in C^1(0, T; S_{hp})$,

b) \[ \left( \frac{\partial u_h(\hat{t})}{\partial \hat{t}}, v_h \right) + C_h(u_h(\hat{t}), v_h) = 0 \quad (5) \]
\[ \forall v_h \in S_{hp}, \; \hat{t} \in (0, T) \]

c) $u_h(0) = u^0_h$, $u^0_h$ is $S_{hp}$-approximation of $u^0$

- semi-discrete problem (5) represents ODEs with IC,
Existence and uniqueness of semi-discrete solution

Theorem
Problem (5) has a unique semi-discrete solution.

Proof (sketch 1/2)
Let \( \{v_j\}_{j=1}^{DOF} \) be the standard base of \( S_{hp} \), i.e.

\[
    u_h(x, \hat{t}) = \sum_{j=1}^{DOF} \xi_j(\hat{t}) v_j(x).
\]

We substitute this into (5), choose and set

\[
    G(\hat{t}) = (\xi_1(\hat{t}), \ldots, \xi_{DOF}(\hat{t}))^T,
\]

\[
    L(\hat{t}) = (\ell_h(v_1)(\hat{t}), \ldots, \ell_h(v_{DOF})(\hat{t}))^T,
\]

\[
    C(\hat{t}) = (c_{ij}(\hat{t}))_{DOF \times DOF}, \ c_{ij}(\hat{t}) = C_h(v_j, v_i).
\]
Theorem

Problem (5) has a unique semi-discrete solution.

Proof (sketch 2/2)

We rewrite (5) as follows: find $G(\hat{t}), \forall \hat{t} \in (0, T)$ such that

$$G'(\hat{t}) = -C(\hat{t}) \cdot G(\hat{t}) + L(\hat{t}),$$

$$G(0) = (\xi_1(0), \ldots, \xi_{DOF}(0))^T \text{ (given by } u_h(0)).$$

Numerical flux $H$ is Lipschitz continuous and form $C_h(\cdot, \cdot)$ is bilinear $\Rightarrow$ RHS of afore-mentioned ODEs is Lipschitz continuous.

The proof is completed by theory of ODEs.
**Time discretization**

- **linearity** of form $C_h(\cdot, \cdot) \Rightarrow$ implicit treatment,
- implicit approach via backward Euler method
- let $0 = t_0 < t_1 < \cdots < t_r = T$ be a partition of $(0, T)$ with constant step $\tau = T/r$, $u_h(t_m) \approx u_h^m \in S_{hp}$, $m = 0, \ldots, r$

**First order implicit scheme**

$$
\left(\frac{1}{\tau} + \gamma\right) (u_h^{m+1}, v_h) + a_h(u_h^{m+1}, v_h) + b_h(u_h^{m+1}, v_h) \\
+ J_h(u_h^{m+1}, v_h) = \frac{1}{\tau} (u_h^m, v_h) + \ell_h(v_h)(t_{m+1}) \quad \forall v_h \in S_{hp},
$$

- problem (6) $\iff$ system of linear algebraic equations.
**Theorem**

Problem (6) has a unique discrete solution.

**Proof (sketch 1/3)**

Suppose $u_h^m \equiv 0$, i.e. $\ell_h(v_h) = 0$, then problem (6) is equivalent to

$$
\left( \frac{1}{\tau} + \gamma \right) (u_h^{m+1}, v_h) + a_h (u_h^{m+1}, v_h) + b_h (u_h^{m+1}, v_h) + J_h (u_h^{m+1}, v_h) = 0
$$

Problem (6) is linear algebraic system $\Rightarrow$ the existence is implied by uniqueness.
Existence and uniqueness of fully discrete solution

**Theorem**

Problem (6) has a unique discrete solution.

**Proof (sketch 2/3)**

Taking \( v_h = u_h^{m+1} \) and neglecting some positive terms, we obtain

\[
\frac{1}{\tau} \| u_h^{m+1} \|^2 \leq |b_h(u_h^{m+1}, u_h^{m+1})| + |\gamma| \| u_h^{m+1} \|^2
\]

Since

\[
|b_h(u_h^{m+1}, u_h^{m+1})| \leq C \left( J_h(u_h^{m+1}, u_h^{m+1})^{1/2} + |u_h^{m+1}|_{1,w} \right) \| u_h^{m+1} \|
\]

then

\[
\| u_h^{m+1} \|^2 \leq \tau C \left( J_h(u_h^{m+1}, u_h^{m+1})^{1/2} + |u_h^{m+1}|_{1,w} \right) \| u_h^{m+1} \|
\]

\[+ \tau |\gamma| \| u_h^{m+1} \|^2\]
Existence and uniqueness of fully discrete solution

**Theorem**

Problem (6) has a unique discrete solution.

**Proof (sketch 3/3)**

For sufficiently small $\tau$, it holds

$$\|u_h^{m+1}\|^2 \leq \delta \|u_h^{m+1}\|^2, \quad \delta \in (0, 1)$$

Then $\|u_h^{m+1}\| = 0$, i.e. $u_h^{m+1} \equiv 0$.

We prove that exists only trivial solution for homogeneous linear algebraic system $\Rightarrow$ existence and uniqueness.
Linear algebraic problem

- standard basis of space $S_{hp}$: $\{v_j\}_{j=1}^{DOF}$,
- linear algebraic representation:

$$\sum_{j=1}^{DOF} \xi_j^m v_j(x) = u_h^m(x) \leftrightarrow U_m = (\xi_1^m, \ldots, \xi_{DOF}^m)^T \in \mathbb{R}^{DOF}$$

- sparse matrix equation

$$\left( \frac{1}{\tau} + \gamma \right) M + A + B + J \right) U_{m+1} = \frac{1}{\tau} M U_m + F_{m+1} \quad (7)$$

- $U_m$ unknown vector, $M$ mass matrix, $A$ stiffness matrix,
- $B$ matrix representing the form $b_h$, $J$ penalty matrix,
- $F_{m+1} = \ell_h(v_h)(t_{m+1})$ right-hand side vector.
Fictional triangulation
Structure of system matrix \( C \)

- \( C_{k,1,1} \): 12x12
- \( C_{k,1,2} \): 12x40
- \( C_{k,2,1} \): 40x12
- \( C_{k,2,2} \): 40x40
- \( C_{k,2,3} \): 40x12
- \( C_{k,2,4} \): 40x24
- \( C_{k,3,2} \): 12x40
- \( C_{k,3,3} \): 12x12
- \( C_{k,4,2} \): 24x40
- \( C_{k,4,4} \): 24x24
- \( C_{k,4,5} \): 24x24
- \( C_{k,4,6} \): 24x24
- \( C_{k,5,4} \): 24x24
- \( C_{k,5,5} \): 24x24
- \( C_{k,5,6} \): 24x12
- \( C_{k,6,5} \): 12x24
- \( C_{k,6,6} \): 12x12
Freefem++ solver

- source: http://www.freefem.org/ff++/
- C++ code and software to solve PDEs numerically,
- user friendly input language allows for a quick specification of any PDE based on a variational formulation,
- easy mesh generation/adaptation,
- available library of a wide range of finite elements,
- implemented sparse solvers,
- basic post-processing/vizualization,
- for more details see [Hecht].
Implementation settings

- adaptively refined grid,
- constant time step $\tau = 1/365$ (i.e. 1 day),
- piecewise $P_1$ (linear) approximations, sparse solver GMRES
Parameter settings

- basket put option with 60% Allianz ($\alpha_1 = 0.6$) and 40% Deutshce Bank ($\alpha_2 = 0.4$),
- current date: September 13, 2011 (Tuesday),
- volatilities $\sigma_1$ and $\sigma_2$ : 4 ways of its determination,
- expiration date $T = 0.257534$ (i.e. 94 days),
- risk-free interest rate $r = 0.01557$ p.a.,
- Pearson linear correlation $\rho = 0.88$,
- two stocks trades at $S_{1}^{\text{ref}} = 59.79$ and $S_{2}^{\text{ref}} = 23.40$,
- reference point $[S_{1}^{\text{ref}}, S_{2}^{\text{ref}}, A^{\text{ref}}]$, where $A^{\text{ref}} = \alpha_1 S_{1}^{\text{ref}} + \alpha_2 S_{2}^{\text{ref}}$
- maximal stock prices $x_1^{\text{max}} = 5.0$ and $x_2^{\text{max}} = 7.0$. 
Determination of volatility

Real market data of individual options:

(A) constant volatilities $\sigma_1 = 0.6392$, $\sigma_2 = 0.9461$,

(B) piecewise constant implied volatility as a function of $S_i$, i.e.

$$\sigma_i = \sigma_i(S_i), \ i = 1,2,$$

(C) piecewise constant implied volatility as a function of $\alpha_i S_i$, i.e.

$$\sigma_i = \sigma_i(\alpha_i S_i), \ i = 1,2,$$

(D) piecewise constant implied volatility as a function of moneyness, i.e.

$$\sigma_i = \sigma_i \left( \frac{\alpha_i S_i}{\alpha_1 S_{1}^{ref} + \alpha_2 S_{2}^{ref}} \exp(-rT) \right), \ i = 1,2.$$
Asian basket with floating strike (1)

- space-time plot of solution with approach (D)
Asian basket with floating strike (2)

- comparison of approaches, isolvalues of solutions, zoom on \((0, 2.50) \times (0, 1.80)\)

\[(A)\]

- 4.08903 (Asian basket)
- 6.15615 (basket)

\[(B)\]

- 6.70615 (Asian basket)
- 8.55314 (basket)

- \(#\mathcal{T}_h \approx 4000\), results at the reference node \([1.3218, 0.5173]\)
Asian basket with floating strike (3)

- comparison of approaches, isolvalues of solutions, zoom on \((0, 2.50) \times (0, 1.80)\)

\[(C)\]

\[
\begin{align*}
&4.08995 \text{ (Asian basket)} \\
&5.52614 \text{ (basket)}
\end{align*}
\]

\[(D)\]

\[
\begin{align*}
&4.08913 \text{ (Asian basket)} \\
&5.70892 \text{ (basket)}
\end{align*}
\]

- \(\#T_h \approx 4000\), results at the reference node \([1.3218, 0.5173]\)
Asian basket with floating strike (4)

- dependency among particular risk sources,
- basket option value at ref. node vs. correlation plot,
- approach (D) for volatilities
1 Motivation

2 Path-dependent multi-asset options

3 Asian two-asset option with floating strike

4 Asian two-asset option with fixed strike
3D PDE model

- continuous arithmetic Asian two-asset basket option with floating strike = the same PDE model as for Asian two-asset basket option with fixed strike,
- for simplicity, we consider the case of **puts** only (the case of calls can be easy derived from put-call parity),
- different payoffs (puts): \( V(S, A, T) = \max(K - A, 0) \)
- different boundary conditions at far-field boundary (puts)
  - no dependency on \( S_1 \) and \( S_2 \) for payoffs, i.e.,
    \[
    \lim_{S_1 \to \infty} \frac{\partial V}{\partial S_1}(S_1, S_2, A, t) = 0, \quad \lim_{S_2 \to \infty} \frac{\partial V}{\partial S_2}(S_1, S_2, A, t) = 0,
    \]
  - zero price for large value of \( A \), i.e.,
    \[
    \lim_{A \to \infty} V(S_1, S_2, A, t) = 0
    \]
Dimensional reduction

- to avoid the higher complexity of 3D model
- change of variables (approach from [Rogers, Shi])

\[
x_1 = \frac{K - A(T - t)/T}{S_1}, \quad x_2 = \frac{K - A(T - t)/T}{S_2}, \quad \hat{t} = T - t
\]

- transformation of price function

\[
u(x, \hat{t}) = \frac{1}{\sqrt{S_1 S_2}} V(S_1, S_2, A, t), \quad \text{where } x = [x_1, x_2]
\]

- transformation of payoffs (= initial conditions)

\[
u(x, 0) = \frac{1}{\sqrt{S_1 S_2}} V(S_1, S_2, A, T) = \max(\sqrt{x_1 x_2} \cdot \text{sgn}(x_1), 0)
\]

- transformation of PDE: homogeneity w.r.t. \(\sqrt{S_1 S_2}\)

\[\rightarrow \text{separation of factor } \sqrt{S_1 S_2} \rightarrow \text{dimensional reduction from 3D to 2D}\]
2D reformulation

- localization on a truncated computational domain $\Omega$,
- convection-diffusion-reaction equation in divergence-free form

We seek $u : Q_T = \Omega \times (0, T) \rightarrow \mathbb{R}$ such that

$$\frac{\partial u}{\partial \hat{t}} - \text{div} \left( D(x) \cdot \nabla u \right) + \nabla \cdot \vec{f}(x, u) + \gamma(x)u = 0 \quad \text{in} \quad Q_T, \quad (8)$$

$$B(u) = 0 \quad \text{on} \quad \partial \Omega,$$

$$u(x, 0) = u^0(x), \quad x \in \Omega,$$

where

$$D(x) \equiv \left\{ D(x)_{kl} \right\}_{k,l=1}^2 = \frac{1}{2} \begin{pmatrix} \sigma_1^2 x_1^2 & \rho \sigma_1 \sigma_2 x_1 x_2 \\ \rho \sigma_1 \sigma_2 x_1 x_2 & \sigma_2^2 x_2^2 \end{pmatrix},$$
2D reformulation

- localization on a truncated computational domain $\Omega$,
- convection-diffusion-reaction equation in divergence-free form

We seek $u : Q_T = \Omega \times (0, T) \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial \hat{t}} - \text{div} \left( D(x) \cdot \nabla u \right) + \nabla \cdot \bar{f}(x, u) + \gamma(x)u = 0 \quad \text{in } Q_T, \quad (8)$$

$$B(u) = 0 \quad \text{on } \partial \Omega,$n$$

$$u(x, 0) = u^0(x), \quad x \in \Omega,$n$$

where $\bar{f}(x, u) = (f_1(x, u), f_2(x, u))^T$ with

$$f_k(x, u) = \left( \frac{\sigma_k^2}{2} + \rho \sigma_1 \sigma_2 + r + \frac{\alpha_1}{T x_1} + \frac{\alpha_2}{T x_2} \right) x_k \cdot u, \quad k = 1, 2$$
2D reformulation

- localization on a truncated computational domain $\Omega$,
- convection-diffusion-reaction equation in divergence-free form

We seek $u : Q_T = \Omega \times (0, T) \rightarrow \mathbb{R}$ such that

$$\frac{\partial u}{\partial \hat{t}} - \text{div} \left( D(x) \cdot \nabla u \right) + \nabla \cdot \vec{f}(x, u) + \gamma(x) u = 0 \quad \text{in } Q_T, \quad (8)$$

$$B(u) = 0 \quad \text{on } \partial \Omega,$$

$$u(x, 0) = u^0(x), \quad x \in \Omega,$$

where

$$\gamma(x) = -2r - \frac{\alpha_1}{T x_1} - \frac{\alpha_2}{T x_2} - \frac{3}{8} \sigma_1^2 - \frac{3}{8} \sigma_2^2 - \frac{9}{4} \rho \sigma_1 \sigma_2$$
2D reformulation

- localization on a truncated computational domain $\Omega$,
- convection-diffusion-reaction equation in divergence-free form

We seek $u : Q_T = \Omega \times (0, T) \rightarrow \mathbb{R}$ such that

$$\frac{\partial u}{\partial \hat{t}} - \text{div} (D(x) \cdot \nabla u) + \nabla \cdot \vec{f}(x, u) + \gamma(x)u = 0 \quad \text{in } Q_T, \quad (8)$$

$$B(u) = 0 \quad \text{on } \partial \Omega,$$

$$u(x, 0) = u^0(x), \quad x \in \Omega,$$

- $\hat{t}$ - time to maturity $\rightarrow$ i.c. given by payoffs $u^0$
- boundary conditions $B(u)$ specified latter
Boundary conditions (1)

- $x_1 \neq 0$ and $x_2 \neq 0 \rightarrow$ domain $\Omega$ is a convex quadrilateral
Boundary conditions (2)

- setting boundary conditions in consistency with the vector field induced by physical fluxes, i.e. \( \left( \frac{\partial f_1}{\partial u}, \frac{\partial f_2}{\partial u} \right) \)
Boundary conditions (3)

- reformulation of original b.c.
  - non-homogeneous Dirichlet b.c. on axes
    \[ x_1 = k_1 x_2 \text{ and } x_2 = k_2 x_2 \quad (0 < k_1 < k_2) \]
    \[
    u(x_1, x_2, \hat{t}) = u_D(x_1, x_2, \hat{t}), \quad \text{on } \Gamma_i, i = 1, 4, \hat{t} > 0,
    \]
    where \( u_D \) is the solution of 1D problem on \( \Gamma_i, i = 1, 4, \)
  - Robin b.c. at infinity
    \[
    \frac{1}{2} u(x_1, x_2, \hat{t}) - x_i \frac{\partial u}{\partial x_i}(x_1, x_2, \hat{t}) = 0 \quad \text{on } \Gamma_i, i = 2, 3, \hat{t} > 0.
    \]
Weak formulation and DG discretization

- standard Sobolev and Lebesgue spaces (no weighted spaces),
- existence and uniqueness of weak solution

\[
\left( \frac{\partial u}{\partial \hat{t}}, v \right) + (\mathcal{L}_t u, v) = 0 \quad \forall \, v \in H^1_0, \text{ a.e. } \hat{t} \in (0, T),
\]

- DG approach analogous to Asian basket with floating strike,
- similar numerical scheme

\[
\left( \frac{1}{\tau} + \gamma \right) (u^{m+1}_h, v_h) + a_h(u^{m+1}_h, v_h) + b_h(u^{m+1}_h, v_h)
+ J_h(u^{m+1}_h, v_h) = \frac{1}{\tau} (u^m_h, v_h) + \ell_h(v_h)(t_{m+1}) \quad \forall \, v_h \in S_{hp}
\]
Parameter settings

- basket put option with 60% Allianz ($\alpha_1 = 0.6$) and 40% Deutshce Bank ($\alpha_2 = 0.4$),
- strike at 40 Euro,
- current date: September 13, 2011 (Tuesday),
- volatilities: $\sigma_1 = 0.6392$ and $\sigma_2 = 0.9461$,
- expiration date $T = 0.257534$ (i.e. 94 days),
- risk-free interest rate $r = 0.01557$ p.a.,
- Pearson linear correlation $\rho = 0.88$,
- two stocks trades at $S_1^{\text{ref}} = 59.79$ and $S_2^{\text{ref}} = 23.40$,
- reference point $[S_1^{\text{ref}}, S_2^{\text{ref}}, A^{\text{ref}}]$, where $A^{\text{ref}} = \alpha_1 S_1^{\text{ref}} + \alpha_2 S_2^{\text{ref}}$
- maximal values $x_1^{\text{max}} = 3.0$ and $x_2^{\text{max}} = 3.0$. 
Asian basket with fixed strike (1)

- space-time plot of solution at maturity
Asian basket with fixed strike (2)

- isolvalues of solutions

\[ \#T_h \approx 4000, \text{ results at the reference node } [0.6690, 1.7094] \]

2.62975 (Asian basket) \hspace{1cm} 3.56022 (basket)
Asian basket with fixed strike (3)

- dependency among particular risk sources,
- basket option value at ref. node vs. correlation plot
**Conclusion**

**Achieved results**

- fundamentals of option pricing,
- market models based on Black-Scholes equation (PDE),
- combination of Asian and basket options,
- 2 approaches of dimensional reduction w.r.t. payoffs
- nonstationary linear convection-diffusion-reaction problem,
- DG space semi-discretization with backward Euler,
- promising numerical results

**Future work**

- to add other realistic experiments with real reference values,
- to investigate the sensitivity measures (Delta, Gamma, etc.),
- to extend the approach to other path-dependent options.
Motivation

Path-dependent multi-asset options
Asian two-asset option with floating strike
Asian two-asset option with fixed strike

Similar approach
Numerical experiments

References


Thank you for your attention