BEM and Parallel Hierarchical Matrices

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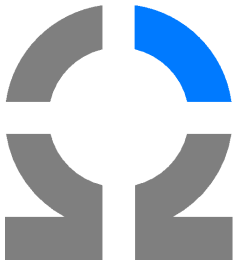

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BEM and Parallel Hierarchical Matrices

Direct method for the Dirichlet problem

Given a lipschitz domain $\Omega \subset \mathbb{R}^2$ and $g \in H^{1/2}(\Gamma)$.

$$\begin{cases} -\triangle u(\mathbf{x}) = 0, & \mathbf{x} \in \Omega \\ u(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \Gamma \end{cases}$$

The first BIE leads to: Find $t := \gamma_{\mathrm{N}} u := \frac{du}{dn} \in H^{-1/2}(\Gamma)$:

$$\langle w, V(t) \rangle = \left\langle w, \left(\frac{1}{2} I + K \right)(g) \right\rangle \quad \forall w \in H^{-1/2}(\Gamma).$$

By Riesz theorem, it is a well–posed problem, provided $\operatorname{diam} \Omega < 1$. The volume solution reads as

$$\begin{aligned} u(\widetilde{\mathbf{x}}) &= [\widetilde{V}(t)](\mathbf{x}) - [W(g)](\mathbf{x}) \\ &\overset{t,g \text{ smooth}}{=} \int_\Gamma \underbrace{-\frac{1}{2\pi} \ln \|\widetilde{\mathbf{x}} - \mathbf{y}\|}_{=:G(\widetilde{\mathbf{x}},\mathbf{y})} t(\mathbf{y})\, ds(\mathbf{y}) - \int_\Gamma \frac{dG}{dn(\mathbf{y})}(\widetilde{\mathbf{x}}, \mathbf{y})\, g(\mathbf{y})\, ds(\mathbf{y}) \end{aligned}$$

BEM and Parallel Hierarchical Matrices

Outline

- Conventional BEM
  - Galerkin boundary element method (BEM)
  - Numerical quadrature of singular kernels
- Fast parallel BEM
  - Fast BEM
  - Parallel BEM
- Applications, conclusion, references

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Galerkin boundary element method (BEM)

2d Galerkin BEM

- Discretization: Decompose the polygonal boundary Γ into disjoint open segments

$$\Gamma = \cup_{i=1}^{n} \overline{S_i}, \quad S_i \cup S_j = \emptyset \text{ for } i \neq j.$$

Sort the segments as well as the end points in the anticlockwise order so that

$$S_i := \{\mathbf{x}(s) := \mathbf{x}_i + (\mathbf{x}_{i+1} - \mathbf{x}_i)\, s : \ 0 < s < 1\}, \quad |S_i| := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|, \quad \text{where } \mathbf{x}_{n+1} := \mathbf{x}_1.$$

- Approximate $H^{-1/2}(\Gamma)$ by L_0^h consisting of piecewise constant functions

$$L_0^h := \langle \Psi_1(\mathbf{x}), \ldots, \Psi_n(\mathbf{x}) \rangle, \quad \text{where } \Psi_i(\mathbf{x}) := \begin{cases} 1 & \mathbf{x} \in S_i, \\ 0 & \text{elsewhere} \end{cases}$$

- Approximate $H^{1/2}(\Gamma)$ by L_1^h consisting of continuous piecewise linear functions

$$L_1^h := \langle \varphi_1(\mathbf{x}), \ldots, \varphi_n(\mathbf{x}) \rangle, \quad \text{where } \varphi_i \in C(\Gamma),\ \varphi_i(\mathbf{x})|_{S_j} = \mathbf{a}_{ij} \cdot \mathbf{x} + b_{ij} \text{ a } \varphi_i(\mathbf{x}_j) = \delta_{ij}$$

Galerkin boundary element method (BEM)

Single–layer matrix V

Recall the 1–layer potential and the formula for $w \in L^\infty(\Gamma)$:

$$V \in \mathcal{L}\left(H^{-1/2}(\Gamma), H^{1/2}(\Gamma)\right), \quad [V(w)](\mathbf{x}) = \lim_{\varepsilon \to 0_+} \int_{\Gamma \setminus B_\varepsilon(\mathbf{x})} w(\mathbf{y})\, G(\mathbf{x},\mathbf{y})\, dl(\mathbf{y}).$$

Thus, for $w(\mathbf{y}), z(\mathbf{x}) \in L_0^h \subset H^{-1/2}(\Gamma)$:

$$\langle z(\mathbf{x}), V(w(\mathbf{y}))\rangle = \left\langle \sum_{i=1}^n z_i \Psi_i(\mathbf{x}), V\left(\sum_{j=1}^n w_j \Psi_j(\mathbf{y})\right)\right\rangle = \mathbf{z}\cdot\mathbf{V}\cdot\mathbf{w},$$

where

$$(\mathbf{V})_{ij} := \int_{S_i}\int_{S_j} G(\mathbf{x},\mathbf{y})\, dl(\mathbf{y})\, dl(\mathbf{x}), \quad \mathbf{z} := (z_1,\ldots,z_n), \quad \mathbf{w} := (w_1,\ldots,w_n).$$

Galerkin boundary element method (BEM)

Double–layer matrix K

Recall the 2–layer potetial and the formula for $v \in L^\infty(\Gamma)$:

$$K \in \mathcal{L}\left(H^{1/2}(\Gamma), H^{1/2}(\Gamma)\right), \quad [K(v)](\mathbf{x}) := \lim_{\varepsilon \to 0_+} \int_{\Gamma \setminus B_\varepsilon(\mathbf{x})} v(\mathbf{y})\, \gamma_{\mathrm{N},\mathbf{y}} G(\mathbf{x},\mathbf{y})\, dl(\mathbf{y}).$$

Thus, for $v(\mathbf{y}) \in L_1^h \subset H^{1/2}(\Gamma)$ and $z(\mathbf{x}) \in L_0^h \subset H^{-1/2}(\Gamma)$:

$$\langle z(\mathbf{x}), K(v(\mathbf{y}))\rangle = \left\langle \sum_{i=1}^n z_i \Psi_i(\mathbf{x}), K\left(\sum_{j=1}^n v_j \varphi_j(\mathbf{y})\right)\right\rangle = \mathbf{z}\cdot\mathbf{K}\cdot\mathbf{v},$$

where

$$(\mathbf{K})_{ij} := \int_{S_i}\int_{S_{j-1}\cup S_j} \varphi_j(\mathbf{y})\, \frac{dG}{d\mathbf{n}_\mathbf{y}}(\mathbf{x},\mathbf{y})\, dl(\mathbf{y})\, dl(\mathbf{x}), \quad \mathbf{z} := (z_1,\ldots,z_n), \quad \mathbf{v} := (v_1,\ldots,v_n),$$

where $S_0 := S_n$.

Galerkin boundary element method (BEM)

Mass matrix M

Yet, for $w, t \in H^{-1/2}(\Gamma)$ and $u, v \in H^{1/2}(\Gamma)$ in BIE there are terms

$$\langle w, I(u)\rangle = \langle w, u\rangle = \int_\Gamma w(\mathbf{x})\, u(\mathbf{x})\, dl(\mathbf{x}), \quad \text{resp. } \langle I(t), v\rangle = \langle t, v\rangle = \int_\Gamma t(\mathbf{x})\, v(\mathbf{x})\, dl(\mathbf{x})$$

Thus, for $u \in L_1^h \subset H^{1/2}(\Gamma)$, $w \in L_0^h \subset H^{-1/2}(\Gamma)$:

$$\langle w, I(u)\rangle = \left\langle \sum_{i=1}^{n} w_i \Psi_i(\mathbf{x}) \sum_{j=1}^{n} u_j \varphi_j(\mathbf{x}) \right\rangle = \mathbf{w} \cdot \mathbf{M} \cdot \mathbf{u},$$

where

$$(\mathbf{M})_{ij} := \int_{S_i} \varphi_j(\mathbf{x})\, dl(\mathbf{x}) = \begin{cases} |S_i|/2 & j = i \text{ or } j = i+1 \text{ or } (i = 1 \text{ and } j = n) \\ 0 & \text{elsewhere} \end{cases}$$

Galerkin boundary element method (BEM)

Dirichlet problem

Given a polygonal domain $\Omega \subset \mathbb{R}^2$ with $\operatorname{diam}\Omega < 1$ and $g \in C(\Gamma)$.

$$\begin{cases} -\triangle u(\mathbf{x}) = 0, & \mathbf{x} \in \Omega \\ u(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \Gamma \end{cases}$$

The Galerkin approximation of the first BIE leads to the linear system

$$\mathbf{V} \cdot \mathbf{t} = \left(\frac{1}{2}\mathbf{M} + \mathbf{K} \right) \cdot \mathbf{g},$$

where $g_i := g(\mathbf{x}_i)$. The Neumann datum is approximated by $t^h(\mathbf{x}) := \sum_{i=1}^{n} t_i \Psi_i(\mathbf{x})$ and for $\mathbf{x} \in \Omega$:

$$u^h(\mathbf{x}) = \sum_{i=1}^{n} t_i \int_{S_i} G(\mathbf{x}, \mathbf{y})\, dl(\mathbf{y}) - \sum_{i=1}^{n} g_i \int_{S_{i-1} \cup S_i} \varphi_i(\mathbf{y}) \frac{dG}{dn_{\mathbf{y}}}(\mathbf{x}, \mathbf{y})\, dl(\mathbf{y}).$$

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Numerical quadrature of singular kernels

Three types of integrals

When evaluating the entries of $\mathbf{V}$ and $\mathbf{K}$, we deal with the following integrals:

a) Identical segments — singularity in $\{\mathbf{x} = \mathbf{y} : \ \mathbf{x}, \mathbf{y} \in S_i\}$, e.g.

$$(\mathbf{V})_{i,i} = \int_{S_i} \int_{S_i} G(\mathbf{x}, \mathbf{y})\, dl(\mathbf{y})\, dl(\mathbf{x}) = -\frac{|S_i|^2}{2\pi} \int_0^1 \int_0^1 \ln(|S^i|\,|s-t|)\, dt\, ds.$$

b) Segments with a common node — singularity at the node, e.g.

$$(\mathbf{V})_{i-1,i} = \int_{S_{i-1}} \int_{S_i} G(\mathbf{x}, \mathbf{y})\, dl(\mathbf{y})\, dl(\mathbf{x}) = -\frac{|S_{i-1}|\,|S_i|}{2\pi} \int_0^1 \int_0^1 \ln \|(\mathbf{x}_{i-1}-\mathbf{x}_i)s-(\mathbf{x}_{i+1}-\mathbf{x}_i)t\|\, dt\, ds.$$

c) Disjoint segments — the kernel is a C^∞ function, e.g.

$$(\mathbf{V})_{i,j} = \int_{S_i} \int_{S_j} G(\mathbf{x}, \mathbf{y})\, dl(\mathbf{y})\, dl(\mathbf{x}) = -\frac{|S_i|\,|S_j|}{2\pi} \int_0^1 \int_0^1 \ln \|\mathbf{x}_i+(\mathbf{x}_{i+1}-\mathbf{x}_i)s-\mathbf{x}_j-(\mathbf{x}_{j+1}-\mathbf{x}_j)t\|\, dt\, d$$

Numerical quadrature of singular kernels

b) Numerical quadrature over segments with a common node

Consider the parameterization $S_{i-1} := \{\mathbf{x} := \mathbf{x}_i + (\mathbf{x}_{i-1} - \mathbf{x}_i)\, s : 0 < s < 1\}$ and $S_i := \{\mathbf{y} := \mathbf{x}_i + (\mathbf{x}_{i+1} - \mathbf{x}_i)\, t : 0 < t < 1\}$

$$(\mathbf{V})_{i-1,i} = \int\limits_{S_{i-1}} \int\limits_{S_i} G(\mathbf{x},\mathbf{y})\, dl(\mathbf{y})\, dl(\mathbf{x}) = -\frac{|S_{i-1}|\,|S_i|}{2\pi} \int\limits_0^1 \int\limits_0^1 \ln \|(\mathbf{x}_{i-1}-\mathbf{x}_i)s-(\mathbf{x}_{i+1}-\mathbf{x}_i)t\|\, dt\, ds.$$

The kernel $k(s,t)$ has a singularity at the origin $s = t = 0$. We replace it by decomposing the integration domain and the Duffy substitution $\tau := s$, $\tau\,\eta := p$

$$\int_0^1 \int_0^1 ds\, dt = \int_0^1 \int_0^t k(s,t)\, ds\, dt + \int_0^1 \int_0^s k(s,t)\, dt\, ds = \int_0^1 \int_0^\tau (k(\tau,p)+k(p,\tau))\, dp\, d\tau$$
$$= \int_0^1 \int_0^1 \tau(k(\tau,\tau\eta) + k(\tau\eta,\tau))\, d\eta\, d\tau.$$

The resulting kernel is continuous and we can employ e.g. a Gauss quadrature

$$(\mathbf{V})_{i,i} \approx \mathbf{w}^{(N)} \cdot \left(\xi_k^{(N)} \left(k(\xi_k^{(N)}, \xi_k^{(N)}\, \xi_l^{(N)}) + k(\xi_k^{(N)}\, \xi_l^{(N)}, \xi_k^{(N)})\right)\right)_{k,l=1}^{N} \cdot \mathbf{w}^{(N)}.$$

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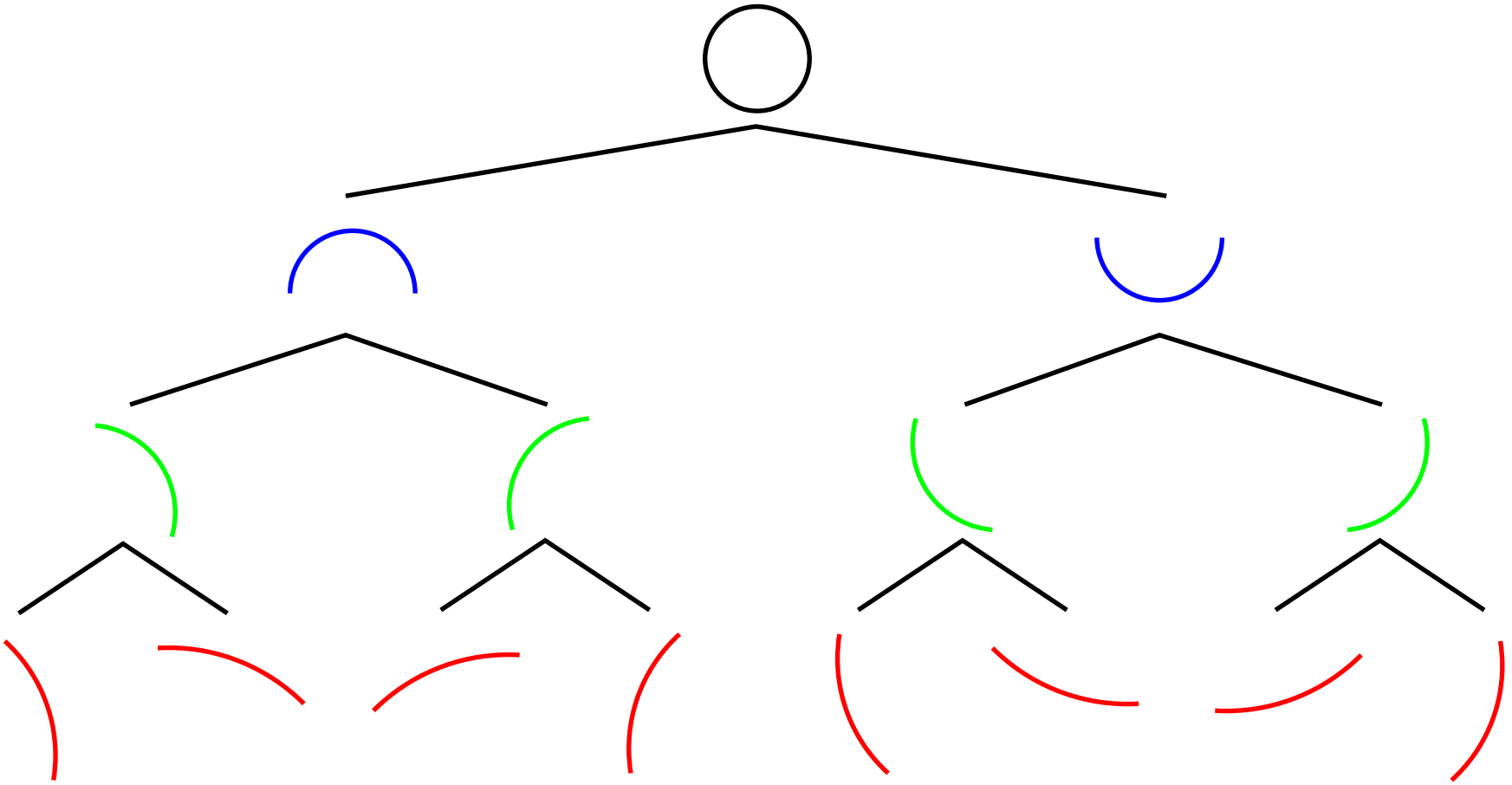
Fast BEM

Hierarchical clustering of the geometry

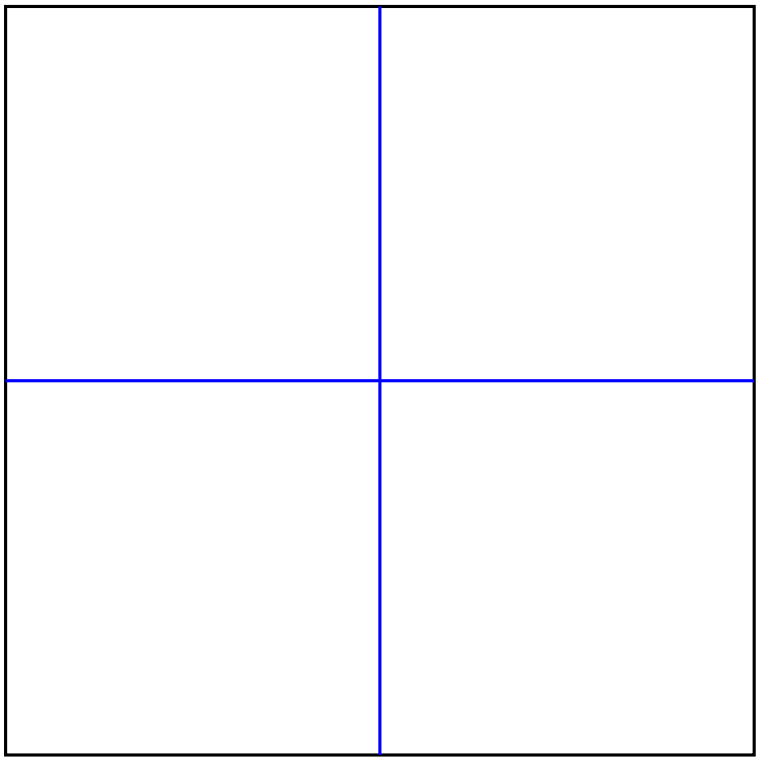


Fast BEM

Near-field and far-field interactions, hierarchical matrices

Clusters C_x and C_y are referred to as an admissible pair if

$$\exists \eta \in (0,1): \ \min\{\operatorname{diam} C_x, \operatorname{diam} C_y\} \leq \eta \operatorname{dist}(C_x, C_y)$$



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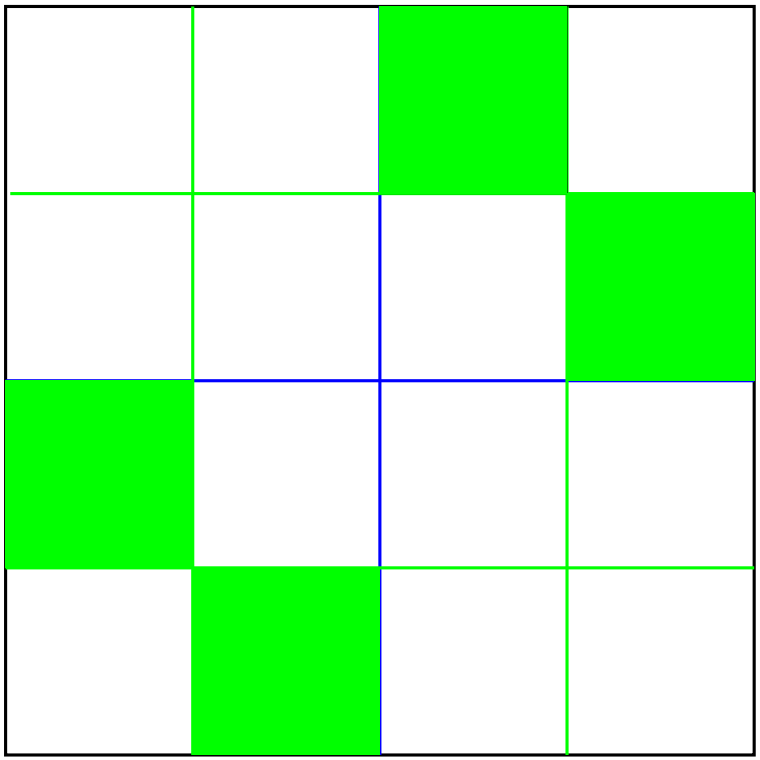

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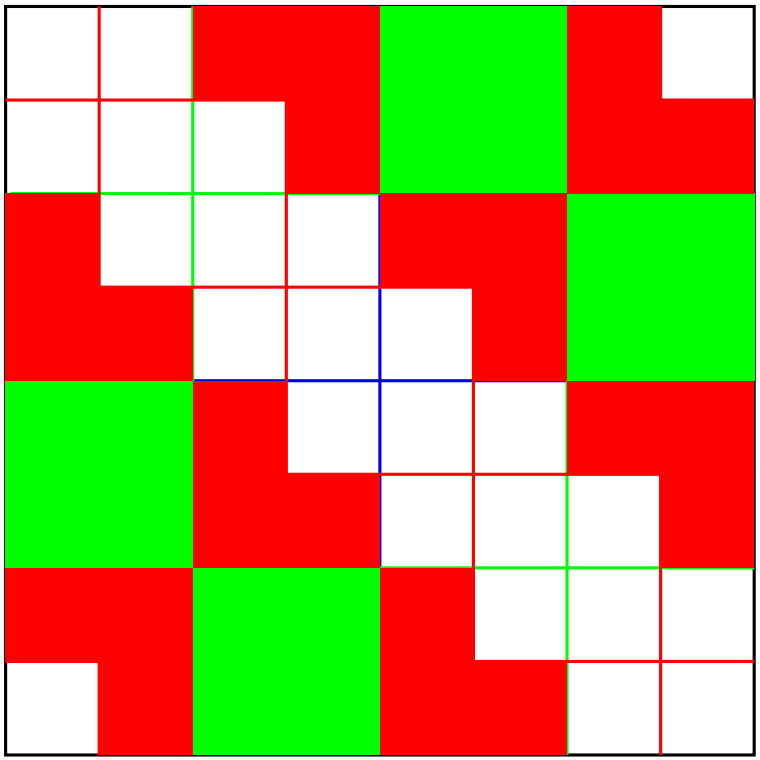


Fast BEM

Adaptive cross approximation (ACA) ≡ skeleton interpolation

$$\mathbf{P}_{C_x}\,\mathbf{A}\,\mathbf{P}_{C_y}^T =: \begin{pmatrix} \widetilde{\mathbf{A}}_{11} & \widetilde{\mathbf{A}}_{12} \\ \widetilde{\mathbf{A}}_{21} & \widetilde{\mathbf{A}}_{22} \end{pmatrix} \approx \begin{pmatrix} \widetilde{\mathbf{A}}_{11} & \widetilde{\mathbf{A}}_{12} \\ \widetilde{\mathbf{A}}_{21} & \widetilde{\mathbf{A}}_{21}\,\widetilde{\mathbf{A}}_{11}^{-1}\,\widetilde{\mathbf{A}}_{12} \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{A}}_{11} \\ \widetilde{\mathbf{A}}_{21} \end{pmatrix} \left[\widetilde{\mathbf{A}}_{11}^{-1} \left(\widetilde{\mathbf{A}}_{11}, \widetilde{\mathbf{A}}_{12} \right) \right]$$
$$=: (\mathbf{u}_1, \ldots, \mathbf{u}_r)\,(\mathbf{v}_1, \ldots, \mathbf{v}_r)^T.$$

The rank $r := r(\varepsilon)$, where $\widetilde{\mathbf{A}}_{11} \in \mathbb{C}^{r\times r}$, is adaptively controlled by ε as follows:

$$\|\mathbf{u}_{k+1}\|_2\,\|\mathbf{v}_{k+1}\|_2 \le \frac{\varepsilon(1-\eta)}{1+\varepsilon}\|\mathbf{A}_k\|_F, \quad \text{where } \mathbf{A}_k := \sum_{m=1}^{k} \mathbf{u}_m \mathbf{v}_m^T$$

which implies, provided $\|\mathbf{R}_{k+1}\|_F \le \eta\|\mathbf{R}_k\|_F$, that $\frac{\|\mathbf{R}_k\|_F}{\|\mathbf{A}\|_F} \le \varepsilon$, where $\mathbf{R}_k := \mathbf{A} - \mathbf{A}_k$. The pivots, stored in $\mathbf{P}_{C_x}$, $\mathbf{P}_{C_y}$, are chosen as to maximize $|\det \widetilde{\mathbf{A}}_{11}^k|$ with a wish to minimize $\|\mathbf{R}_k\| \equiv \|\widetilde{\mathbf{A}}_{22}^k - \widetilde{\mathbf{A}}_{21}^k\,(\widetilde{\mathbf{A}}_{11}^k)^{-1}\,\widetilde{\mathbf{A}}_{12}^k\|$.

Fast BEM

ACA algorithm: an example ($\mathbf{R}_0 := \mathbf{A}$)

$$\mathbf{R}_0 = \begin{pmatrix} 0.431 & 0.354 & 0.582 & 0.417 \\ 0.491 & 0.396 & 0.674 & 0.449 \\ 0.446 & 0.358 & 0.583 & 0.413 \\ 0.380 & 0.328 & 0.557 & 0.372 \end{pmatrix} \xrightarrow[R=\{1\}]{i_1=1,\, j_1=3} \frac{1}{0.582} \begin{pmatrix} 0.582 \\ 0.674 \\ 0.583 \\ 0.557 \end{pmatrix} (0.431, 0.354, 0.582, 0.417)$$

$$\mathbf{R}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.008 & -0.014 & 0 & -0.034 \\ 0.014 & 0.003 & 0 & -0.005 \\ -0.033 & -0.011 & 0 & -0.027 \end{pmatrix} \xrightarrow[R=\{1,2\}]{i_1=2,\, j_1=4} \frac{1}{-0.034} \begin{pmatrix} 0 \\ -0.034 \\ -0.005 \\ -0.027 \end{pmatrix} (-0.008, -0.014, 0, -0.034)$$

$$\mathbf{R}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.015 & 0.005 & 0 & 0 \\ -0.026 & 0.0004 & 0 & 0 \end{pmatrix} \xrightarrow[R=\{1,2,4\}]{i_1=4,\, j_1=1} \frac{1}{-0.026} \begin{pmatrix} 0 \\ 0 \\ 0.015 \\ -0.026 \end{pmatrix} (-0.026, 0.0004, 0, 0)$$

The relative error decays as follows: $\|\mathbf{R}_k\|_2 / \|\mathbf{A}\|_2 = 0.030, 0.016, 0.003$ for $k = 1, 2, 3$

Fast BEM

Compression of asymptotically smooth functions by Taylor's expansion

Consider an admissible pair of clusters $C_x, C_y \subset \mathbb{R}$, i.e.,

$$\exists \eta \in (0,1): \ \operatorname{diam} C_y \leq \eta \operatorname{dist}(C_x, C_y),$$

an asymptotically smooth $f : C_x \times C_y \to \mathbb{R}$, i.e.,

$$\exists c_1, c_2 > 0 \ \exists g \geq 0 \ \forall p \in \mathbb{N} \ \forall t \in C_y : \quad |f_t^{(p)}(s,t)| \leq c_1 (c_2)^p \, p! / |s-t|^{g+p},$$

and its Taylor's expansion

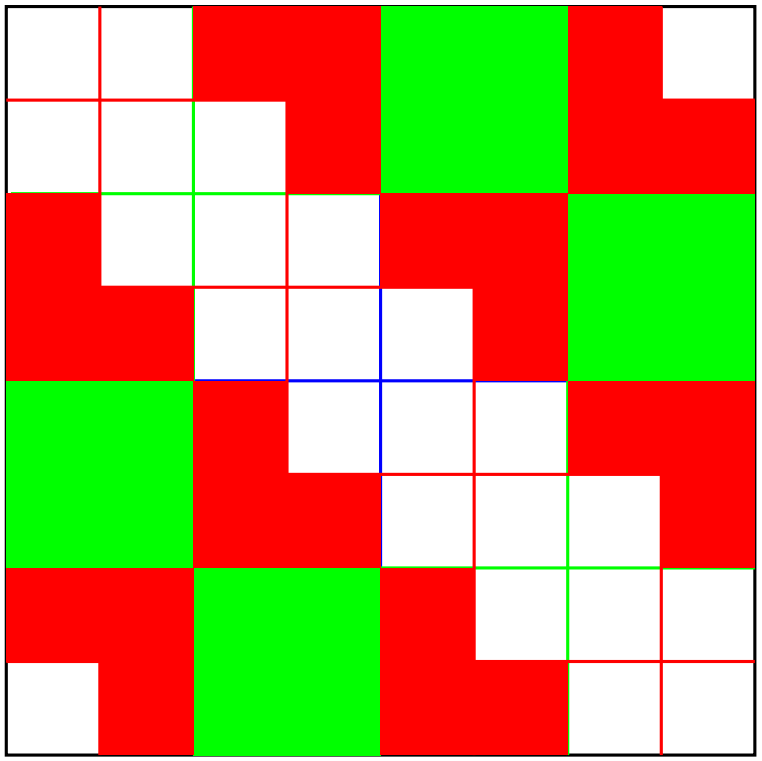
$$f(s,t) = \underbrace{\sum_{k=0}^{p-1} \frac{1}{k!} f_t^{(k)}(s,t_0)(t-t_0)^k}_{\text{rank } p} + \underbrace{\frac{1}{p!} f_t^{(p)}(s,\xi)(t-t_0)^p}_{=:R_p(s,t)} \quad \text{for } |\xi - t_0| < |t - t_0|.$$

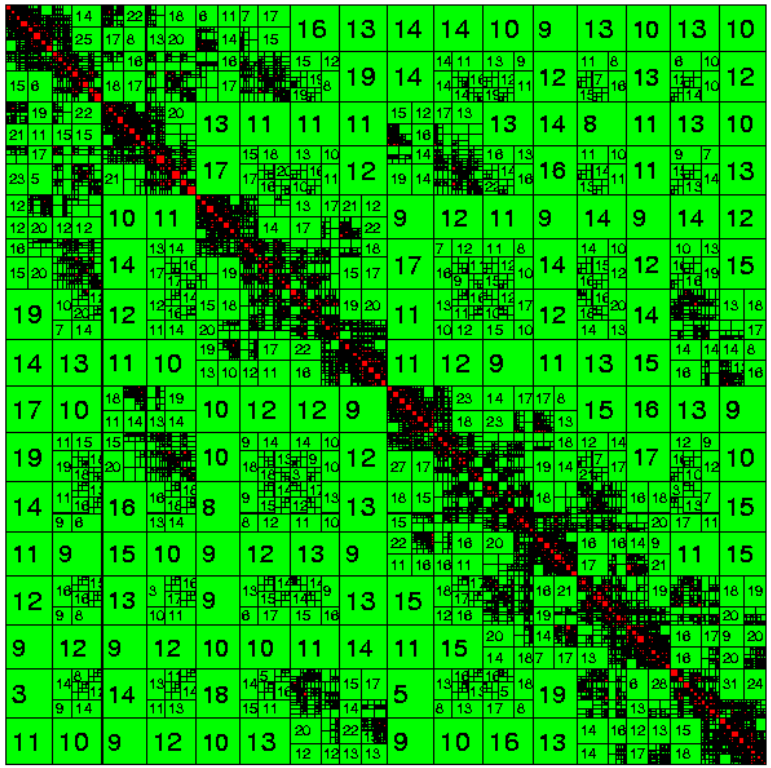
Then

$$|R_p(s,t)| \leq c_1 c_2^p \underbrace{\left|\frac{t-t_0}{s-t}\right|^p}_{\leq \eta^p} \operatorname{dist}^{-g}(C_x, C_y) \leq c_1 (\underbrace{c_2 \eta}_{<1})^p \operatorname{dist}^{-g}(C_x, C_y) \leq \varepsilon \ \Rightarrow \ p \approx \frac{\log \varepsilon}{\log \eta}$$

Fast BEM

$$\mathrm{CPU}, \mathrm{Mem}(\mathbf{A}, \varepsilon) = O(n \, \log n \, \log \varepsilon)$$





$$\mathrm{CPU}, \mathrm{Mem}(\mathbf{A}, \varepsilon) = \underbrace{kn}_{\text{dense blocks}} + \sum_{l=2}^{\log_2 n} 2^l \, 3 \, \underbrace{(2n/2^l) \, p(\varepsilon)}_{\text{compressed blocks}} = O(n \, \log n \, \log \varepsilon)$$

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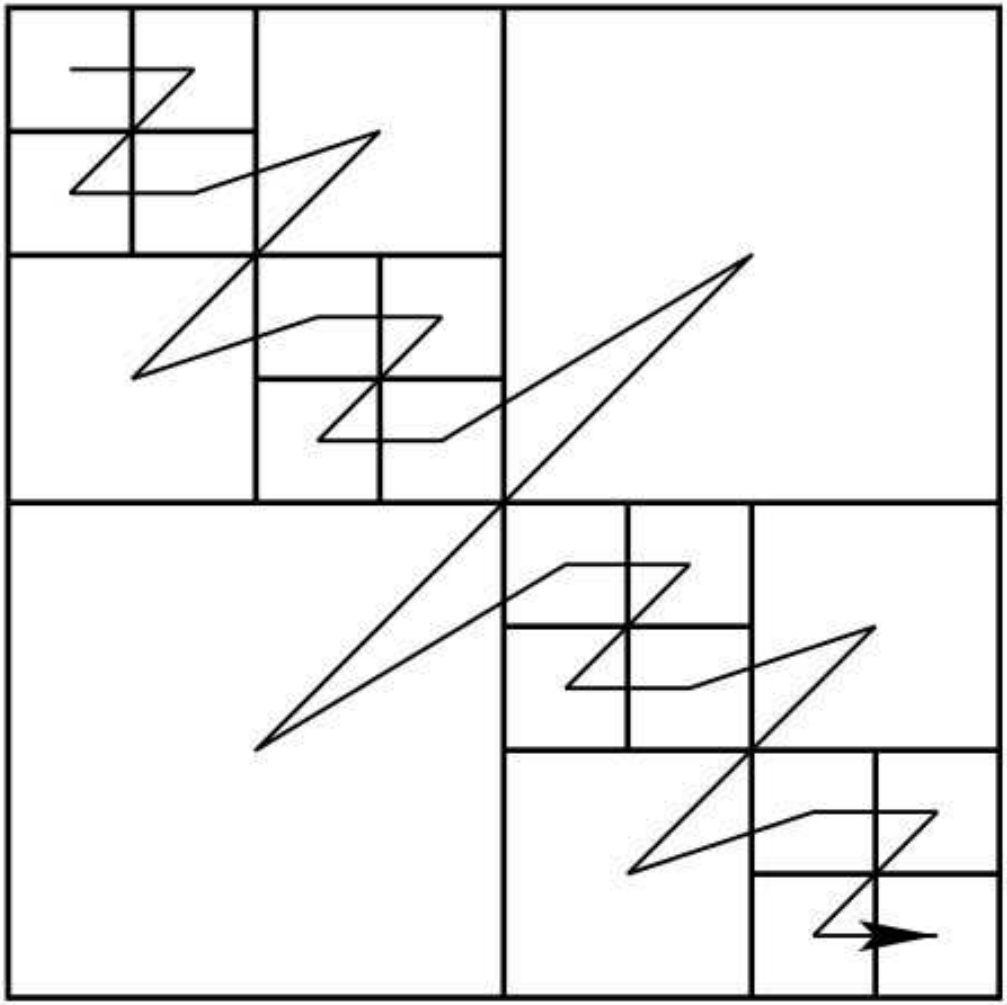
Parallel BEM

Space-filling curves approach [Bebendorf & Kriemann, 2005]



Parallel BEM

Our approach

N processes, $N \times N$ submatrices

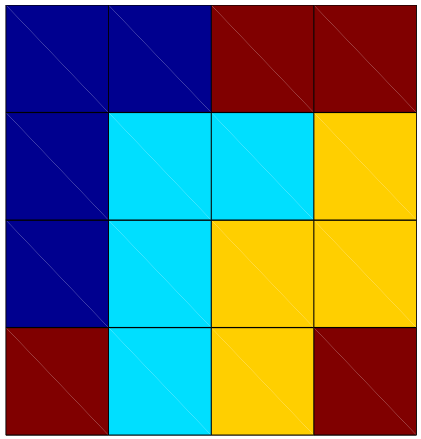
- Each diagonal block with the related geometry data assigned to one process

  $\Rightarrow$ both memory and CPU balanced, since most nonadmissible blocks are distributed efficiently.

- Each geometrically closely related $N-1$ off–diagonal blocks assigned to one process

  ? memory balanced: $Mem = O\left(\frac{n \log n}{N} + \frac{n}{\sqrt{N}}\right)$

  ? CPU balanced



Parallel BEM

Finding optimal distributions by brute force fails

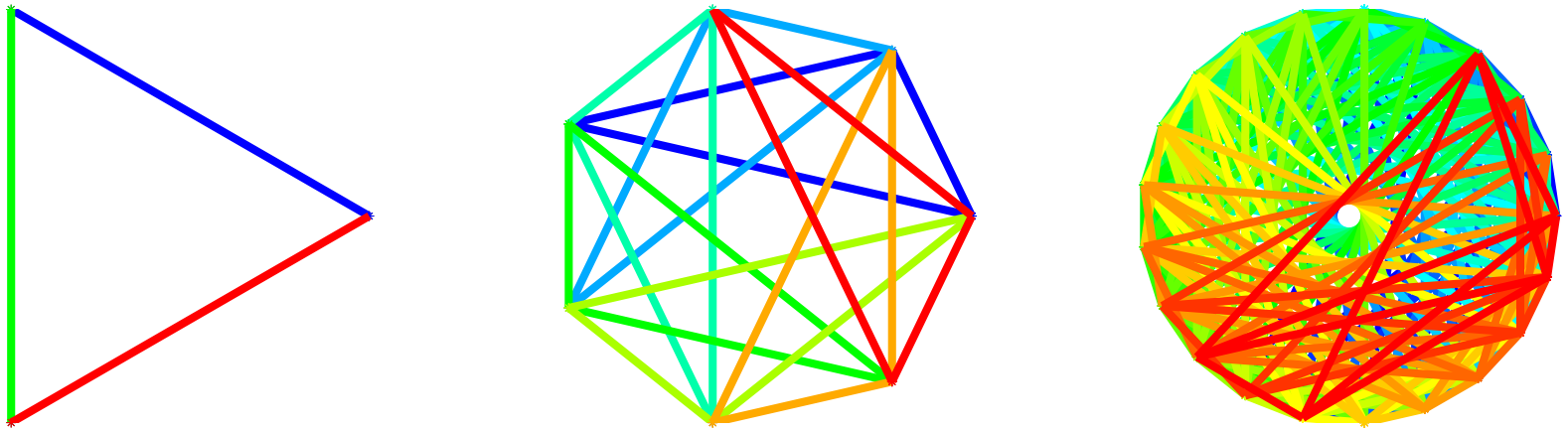
- $N = 2$: 2 cases,
- $N = 4$: 34650 cases,
- $N = 8$: $4 \cdot 10^{42}$ cases.

$$\text{number of cases} = \binom{(N-1)\,N}{N} \cdot \binom{(N-2)\,N}{N} \cdots \binom{2\,N}{N}$$

Parallel BEM

Cyclic decomposition of undirected graphs

$N := 3$ $N := 7$ $N := 21$



It is equivalent to perfect difference sets [Singer, 1934]: decompositions available for

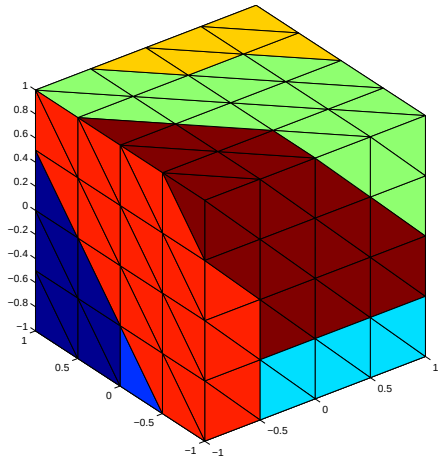
$$\frac{N(N-1)}{2N} = \frac{p(p-1)}{2},$$

where $p+1$ is a power of a prime number.

Parallel BEM

The algorithm

1. Decomposition of the mesh into N submeshes (by Metis).



2. Assignment of $O(\sqrt{N})$ submeshes to each processor, using the cyclic decomposition.
3. Parallel assembling of the $N \times N$ block matrix by means of a fast BEM.

Parallel BEM

ACA for Laplace 1-layer matrix on a cube

n	compr. $\mathbf{V}$	average memory [MB], CPU [s] per process $N := 1$	$N := 7$	$N := 31$	$N := 57$	$N := 73$	$N := 91$	$N := 133$
3072	21.5%	160, 8	148, 1	170, 0	194, 0	177, 0	197, 0	?, 0
12288	13.1%	267, 59	163, 7	175, 1	176, 1	167, 1	200, 1	207, 1
49152	5.2%	884, 367	263, 51	201, 10	194, 8	195, 6	214, 5	220, 4
196608	1.8%		705, 226	353, 53	274, 32	254, 25	280, 25	276, 18
786432	0.7%			999, 294	668, 172	599, 119	570, 110	535, 99
3145728	0.3%							1911 MB, 596 s

ACA: $\eta := 1.1$, $\varepsilon := 10^{-4}, \ldots, 10^{-9}$, $n_{\min} := 10, \ldots, 60$

Parallel scalability: $CPU = O\left(\frac{n \log n}{N}\right), Mem = O\left(\frac{n \log n}{N} + \frac{n}{\sqrt{N}}\right).$

Parallel BEM

Int. Laplace problem with Dir. datum $u(\mathbf{x}) := 1/|\mathbf{x}-(2,2,2)|$ **on** $\Omega := (0,1)^3$

#elems error, #CG	assemble time: CPU($\mathbf{V}$)/CPU($\mathbf{K}$) [s] memory [MB] per process: compression of $\mathbf{V}$/compression of $\mathbf{K}$ [%] $N := 7$	$N := 31$	$N := 57$	$N := 73$	$N := 133$
3072 2.6e-2, 59	114:2/84 159:41/84	42:2/24 173:40/93	24:0/21 176:42/99	20:1/17 192:46/100	
12288 1.3e-2, 78	545:11/396 247:19/41	153:2/81 213:19/45	95:0/54 210:18/49	77:2/75 206:20/53	47:0/30 202:23/67
49152 6.5e-3, 102	2752:69/2209 803:8/16	819:13/474 347:8/17	601:6/280 291:8/19	446:8/292 277:8/20	241:7/176 258:9/25
196608 3.3e-3, 129		3171:83/2521 1025:3/6	2122:45/1282 717:3/7	1885:39/1348 646:3/7	1016:31/790 529:3/8
786432 1.7e-3, 167					4247 s:161/4085 1885 MB:1/3

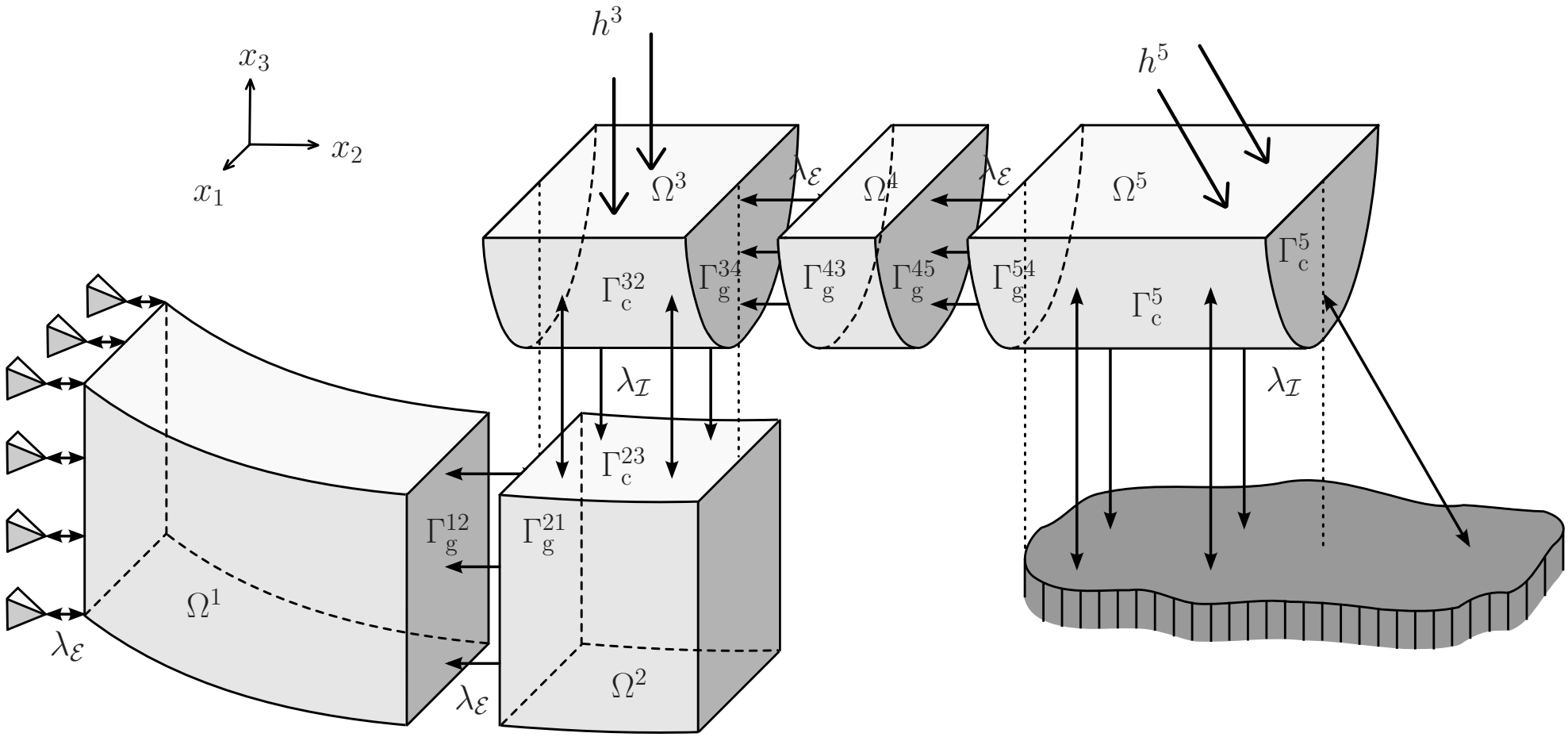
BEM and Parallel Hierarchical Matrices

Outline

- Conventional BEM
  - Galerkin boundary element method (BEM)
  - Numerical quadrature of singular kernels
- Fast parallel BEM
  - Fast BEM
  - Parallel BEM
- Applications, conclusion, references

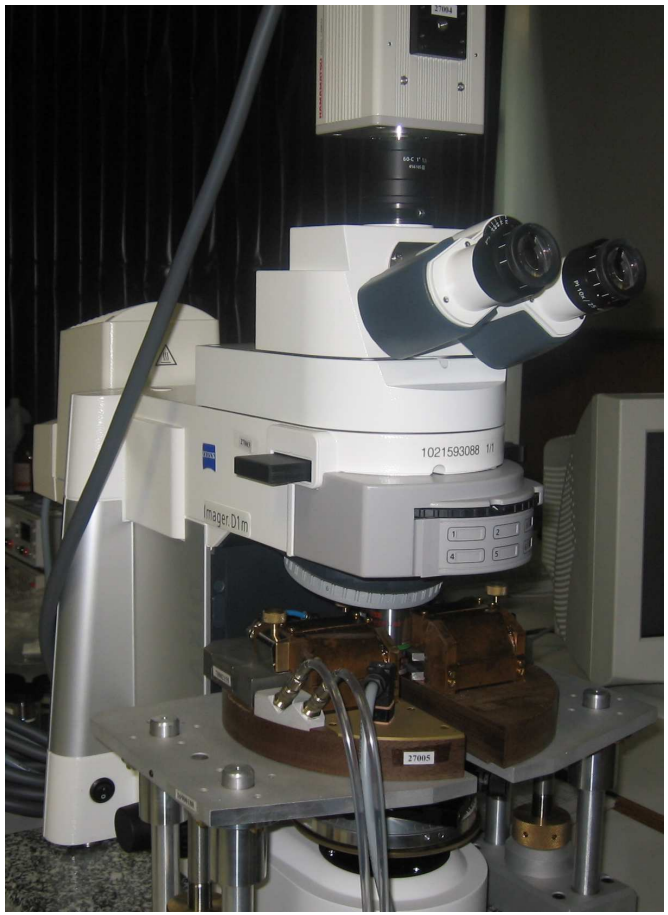
Applications

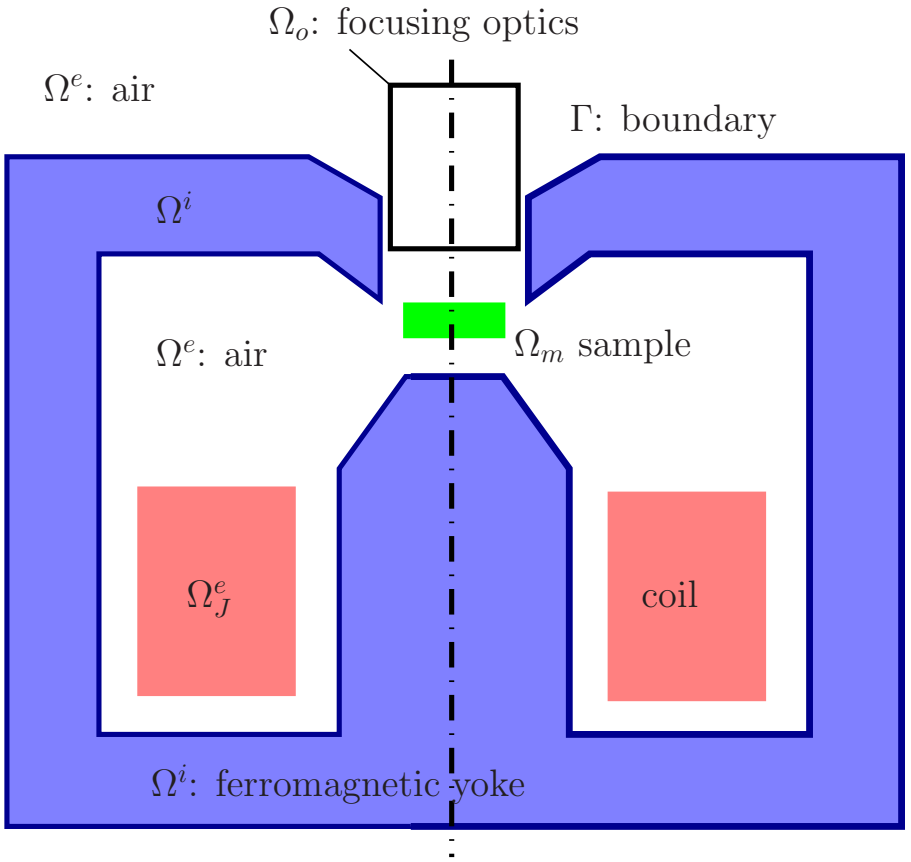
TBETI for contact mechanics [Sadowská, Bouchala, Dostál]



Applications

Shape optimization of a DC electromagnet, FEM-BEM coupling

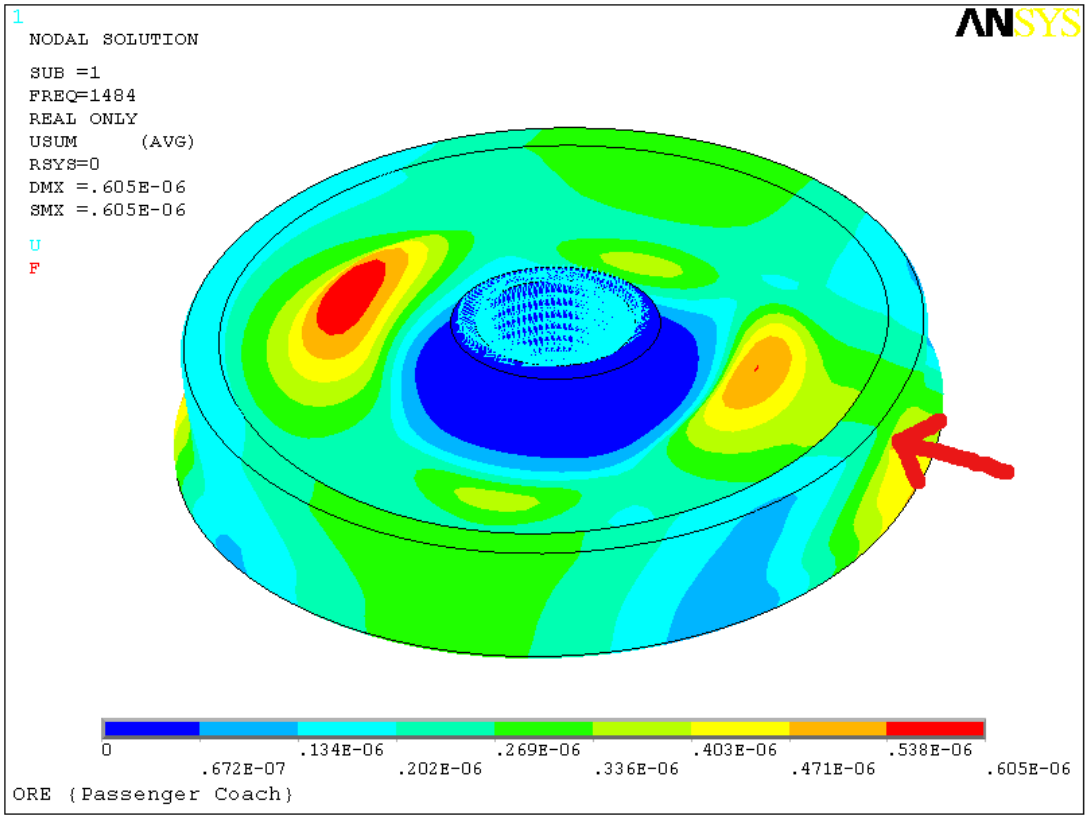




L., Postava, Životský: J Magn Magn Mater '10, Math Comput Simulat '12

Applications

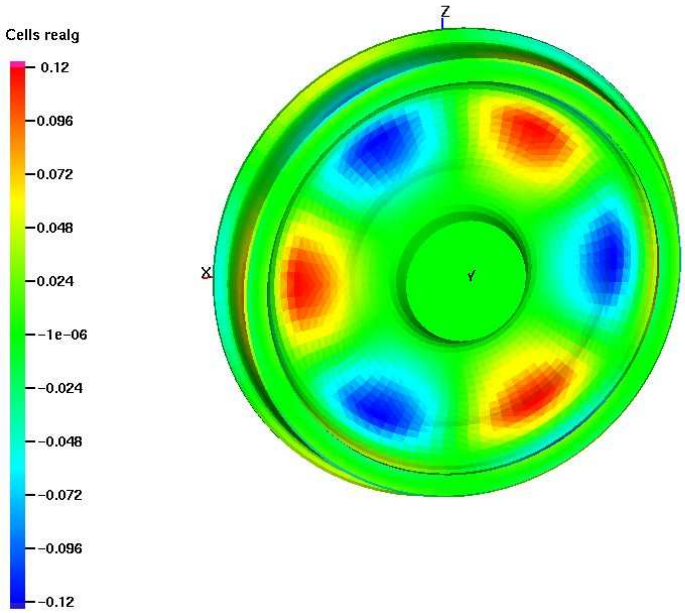
Acoustics of a railway wheel

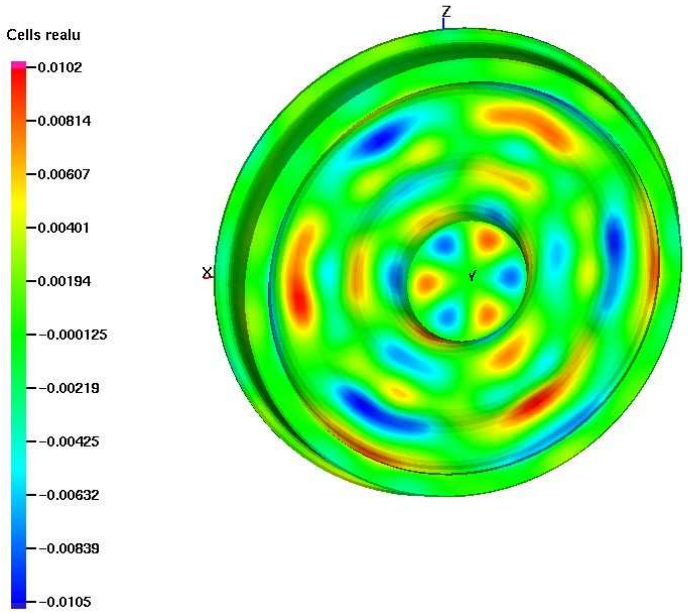


A joint work with J. Szweda, Department of mechanics, VŠB–TU Ostrava

Applications

Acoustics of a railway wheel

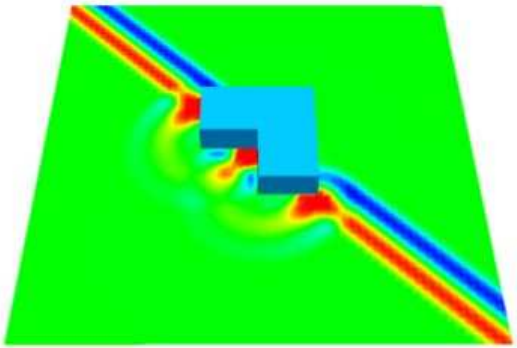
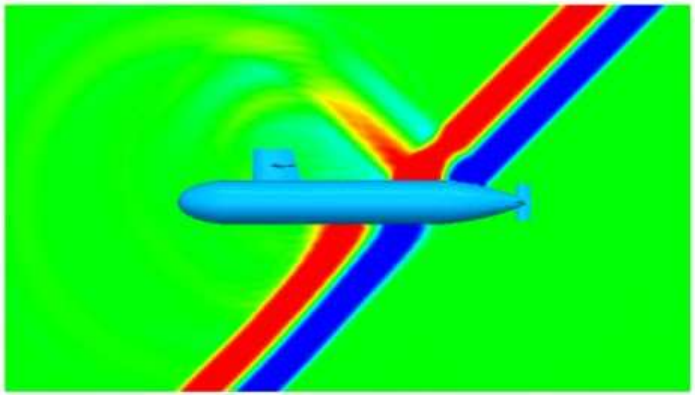
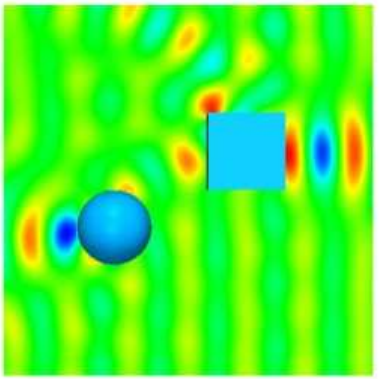




Conclusion, references

Area of use

- BEM reduces the problem to the boundary
- Fundamental solution is known for many 2d/3d PDEs, e.g., elasticity, acoustics, electromagnetism
- Recently also time–domain BEM for parabolic and hyperbolic PDEs
- Problems in bounded as well as unbounded domains
- Natural coupling with FEM
- Cons: restricted to linear material laws, difficult implementation and theory

Conclusion, references

References

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http://homel.vsb.cz/~luk76