Návrh robustního experimentu založený na globální citlivosti

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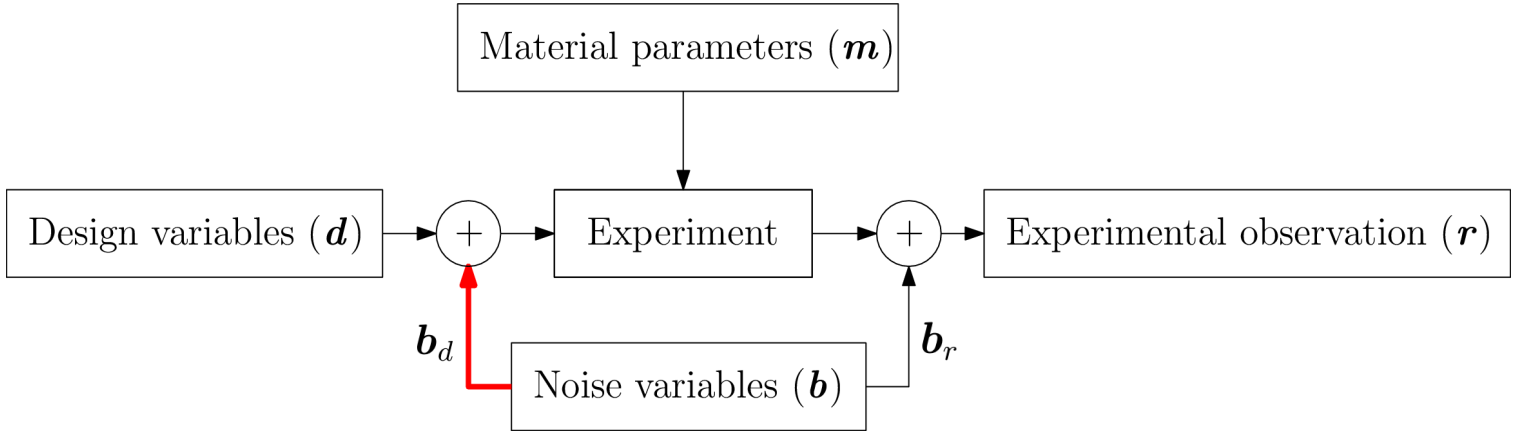
Experiment design problem



Material parameters to be identified:

- $\boldsymbol{m}$ - values of thermal conductivity, heat capacity or water vapour resistance

Design variables to be optimised:

- $\boldsymbol{d}$ - loading types (e.g. prescribed temperature, moisture flux or heat transfer at a boundary) and loading magnitude as well as positions of sensors (thermometers, hygrometers etc.)

Noise variables to be considered: $\boldsymbol{b} = (\boldsymbol{b}_d, \boldsymbol{b}_r)$

- $\boldsymbol{b}_r$ - measurement errors,
- $\boldsymbol{b}_d$ - inaccurate positioning of sensors or imperfections in prescribed loading conditions

Inverse analysis

$$\boldsymbol{r}(\boldsymbol{d}) = \boldsymbol{f}(\boldsymbol{d}, \boldsymbol{b}_d, \boldsymbol{m}) + \boldsymbol{b}_r$$

After realisation of an experiment according to selected values of $\boldsymbol{d}$

Nonlinear regression with random parameters $\boldsymbol{b}_d$

$$\widehat{\boldsymbol{m}} = \underset{\boldsymbol{m} \in \mathscr{M}}{\operatorname{argmin}} \sum_i \left(r_i - f_i(\boldsymbol{d}, \boldsymbol{b}_d, \boldsymbol{m})\right)^2$$

- random = noise parameters follow prescribed probability distribution corresponding to distribution of their values in a set of repeated experiments
- assumption of more repeated experiments

Bayesian inference

$$p(\boldsymbol{m}, \boldsymbol{b}_d | \boldsymbol{r}) \approx p(\boldsymbol{r} | \boldsymbol{m}, \boldsymbol{b}_d) p(\boldsymbol{m}, \boldsymbol{b}_d) = L(\boldsymbol{r} - \boldsymbol{f}(\boldsymbol{d}, \boldsymbol{m}, \boldsymbol{b}_d)) p(\boldsymbol{m}, \boldsymbol{b}_d)$$

- distribution of noise parameters is updated along with the distribution of material properties
- assumption of only one realisation of the experiment, no repetitions

Goals of experiment design

- **Maximise information content on the investigated material parameters**
- **Minimise influence of experimental errors**

$$\max_{\boldsymbol{d}} \quad \mathbf{S}_{\boldsymbol{m}}\,, \qquad \text{(sensitivity of observations to material properties)}$$

$$\min_{\boldsymbol{d}} \quad \mathbf{S}_{\boldsymbol{b}}\,, \qquad \text{(sensitivity of observations to noise variables)}$$

$$\text{s.t.} \quad \boldsymbol{m} \in \mathscr{M}, \boldsymbol{b} \in \mathscr{B}, \boldsymbol{d} \in \mathscr{D}$$

(expert on material model specifies feasible intervals)

(expert on experimentation specifies feasible intervals for design variables and mean and variance of noise variables)

Bayesian approach

- maximising Kullback-Leibner divergence
- quantifying information content of experiment by comparing prior and posterior distribution
- computationally very exhaustive

Nonlinear regression with random parameters

[Ruffio,Saury,Petit, 2012, IJHMT]

$$\boldsymbol{r}(\boldsymbol{d}) = \boldsymbol{f}(\boldsymbol{d}, \boldsymbol{b}_d, \boldsymbol{m}) + \boldsymbol{b}_r$$

$$\widehat{\boldsymbol{m}} = \underset{\boldsymbol{m} \in \mathscr{M}}{\operatorname{argmin}} \sum_i \left(r_i - f_i(\boldsymbol{d}, \boldsymbol{b}_d, \boldsymbol{m})\right)^2$$

Local sensitivity matrix – 1st order derivatives in Taylor development of the model:

$$\boldsymbol{f}_{\overline{\boldsymbol{b}_d}, \widehat{\boldsymbol{m}}}(\boldsymbol{d}, \boldsymbol{b}_d, \boldsymbol{m}) = \boldsymbol{f}(\boldsymbol{d}, \overline{\boldsymbol{b}_d}, \widehat{\boldsymbol{m}}) + \mathbf{S}_{\boldsymbol{b}_d}(\boldsymbol{b}_d - \overline{\boldsymbol{b}_d}) + \mathbf{S}_{\boldsymbol{m}}(\boldsymbol{m} - \widehat{\boldsymbol{m}})$$

Explicit solution of a least square problem:

$$\widehat{\boldsymbol{m}}(\boldsymbol{d}, \boldsymbol{b}_d, \boldsymbol{b}_r) = \left(\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}} \mathbf{S}_{\boldsymbol{m}}\right)^{-1} \mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}} \left[\boldsymbol{r}(\boldsymbol{d}) - \boldsymbol{f}(\boldsymbol{d}, \overline{\boldsymbol{b}_d}, \widehat{\boldsymbol{m}}) - \mathbf{S}_{\boldsymbol{b}_d}(\boldsymbol{b}_d - \overline{\boldsymbol{b}_d}) + \mathbf{S}_{\boldsymbol{m}} \widehat{\boldsymbol{m}}\right]$$

$$\operatorname{cov}(\widehat{\boldsymbol{m}}) = \left(\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}} \mathbf{S}_{\boldsymbol{m}}\right)^{-1} \mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}} \left[\operatorname{cov}(\boldsymbol{b}_r) + \mathbf{S}_{\boldsymbol{b}_d} \operatorname{cov}(\boldsymbol{b}_d) \mathbf{S}_{\boldsymbol{b}_d}^{\mathrm{T}}\right] \mathbf{S}_{\boldsymbol{m}}$$

With neglecting b_d and considering measurement errors to be i.i.d. variables with variance $\sigma_{b_r}^2$: $\operatorname{cov}(\widehat{\boldsymbol{m}}) = \sigma_{b_r}^2 \left(\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}} \mathbf{S}_{\boldsymbol{m}}\right)^{-1}$

Optimality criterion:

F-optimality: $F(\boldsymbol{d}, \widehat{\boldsymbol{m}}, \overline{\boldsymbol{b}}) = \sqrt{\operatorname{trace}\left((\operatorname{cov}(\widehat{\boldsymbol{m}}) \operatorname{diag}(\widehat{\boldsymbol{m}})^{-1})^{\mathrm{T}} (\operatorname{cov}(\widehat{\boldsymbol{m}}) \operatorname{diag}(\widehat{\boldsymbol{m}})^{-1})\right)} = \sqrt{\sum_i \left(\frac{\widehat{\sigma_{m_i}}}{\widehat{m_i}}\right)^2}$

Robustness of solution

[Ruffio,Saury,Petit, 2012, IJHMT]

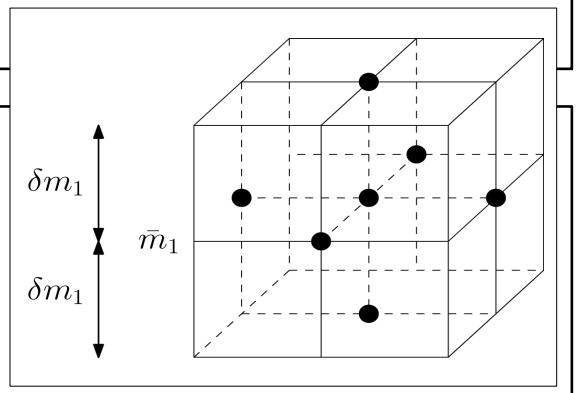
$$\boldsymbol{f}_{\overline{\boldsymbol{b}_d},\widehat{\boldsymbol{m}}}(\boldsymbol{d},\boldsymbol{b}_d,\boldsymbol{m}) = \boldsymbol{f}(\boldsymbol{d},\overline{\boldsymbol{b}_d},\widehat{\boldsymbol{m}}) + \mathbf{S}_{\boldsymbol{b}_d}(\boldsymbol{b}_d - \overline{\boldsymbol{b}_d}) + \mathbf{S}_{\boldsymbol{m}}(\boldsymbol{m} - \widehat{\boldsymbol{m}})$$

$$\mathbf{S}_{\boldsymbol{b}_d} = \mathbf{S}_{\boldsymbol{b}_d}(\boldsymbol{d},\overline{\boldsymbol{b}_d},\widehat{\boldsymbol{m}}) = \begin{bmatrix} \frac{\partial f_1(\cdot)}{\partial b_{d,1}} & \frac{\partial f_1(\cdot)}{\partial b_{d,2}} & \cdots \\ \frac{\partial f_2(\cdot)}{\partial b_{d,1}} & \frac{\partial f_2(\cdot)}{\partial b_{d,2}} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad \mathbf{S}_{\boldsymbol{m}} = \mathbf{S}_{\boldsymbol{m}}(\boldsymbol{d},\overline{\boldsymbol{b}_d},\widehat{\boldsymbol{m}}) = \begin{bmatrix} \frac{\partial f_1(\cdot)}{\partial m_1} & \frac{\partial f_1(\cdot)}{\partial m_2} & \cdots \\ \frac{\partial f_2(\cdot)}{\partial m_1} & \frac{\partial f_2(\cdot)}{\partial m_2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Requires starting guess about $\widehat{\boldsymbol{m}}$ which is apriori unknown!

Worst case approach

- Noise parameters $\boldsymbol{b}$ fixed at mean (zero) values
- Material parameters $\boldsymbol{m}$ evaluated in central star design points



$$F_M(\boldsymbol{d}) = \max\{F(\bar{\boldsymbol{m}},\boldsymbol{d}), F(\bar{\boldsymbol{m}} - \delta\boldsymbol{m}_i\boldsymbol{e}_i,\boldsymbol{d}), F(\bar{\boldsymbol{m}} + \delta\boldsymbol{m}_i\boldsymbol{e}_i,\boldsymbol{d}) : 1 \le i \le N_m\}$$

Robust optimisation of $F_M(\boldsymbol{d})$ by particle swarm optimisation algorithm.

Formulation of using global sensitivity matrix

Polynomial chaos-based approximation of the model

$$\boldsymbol{r}(\boldsymbol{d}) = \boldsymbol{f}(\boldsymbol{d}, \boldsymbol{b}, \boldsymbol{m})$$

All problem variables need to be transformed into the standardised variables of a chosen family via linear or nonlinear transformation

$$(\boldsymbol{d}, \boldsymbol{b}, \boldsymbol{m}) = \boldsymbol{g}(\boldsymbol{\xi} = (\boldsymbol{\xi}_{\boldsymbol{d}}, \boldsymbol{\xi}_{\boldsymbol{b}}, \boldsymbol{\xi}_{\boldsymbol{m}}))$$

According to Doob-Dynkin lemma (Bobrovski, 2005), model response is also random vector expressed as a function of the same random variables.

$$\widetilde{\boldsymbol{f}}(\boldsymbol{\xi}) = \sum_{\alpha \in \mathcal{J}} \boldsymbol{f}_{\alpha} H_{\alpha}(\boldsymbol{\xi})$$

According to (Xiu and Karniadakis, 2002, SIAM JSC), orthogonal polynomials w.r.t. given probability measures are the most suitable choice for the approximation.

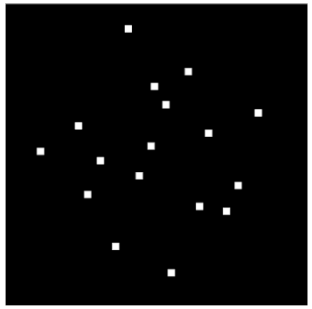
- Hermite polynomials for Gaussian variables
- Legendre polynomials for uniform variables etc.

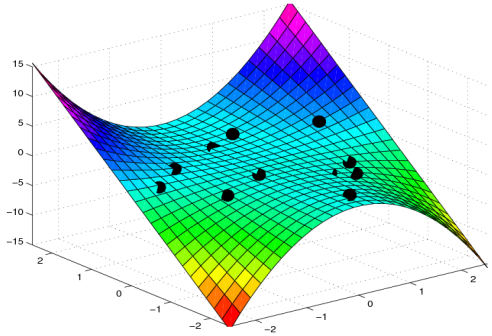
Formulation of using global sensitivity matrix

Construction of PCE-based approximation – computation of PC coefficients

- Linear regression
- Stochastic collocation method
- Stochastic Galerkin method

We use linear regression based on Latin Hypercube sampling derived from Halton sequence - very easy to implement





Global sensitivity matrix – variance-based Sobol' indices

$$S^*_{f_k,\xi_i} = \frac{\sum_{\boldsymbol{\alpha}\in\mathcal{I}_i} f^2_{k,\boldsymbol{\alpha}}\mathbb{E}[H^2_{\boldsymbol{\alpha}}(\boldsymbol{\xi})] + \sum_{\boldsymbol{\alpha}\in\mathcal{I}^*_i} {}^1\!/_{n^*_i} f^2_{k,\boldsymbol{\alpha}}\mathbb{E}[H^2_{\boldsymbol{\alpha}}(\boldsymbol{\xi})]}{\sum_{\boldsymbol{\alpha}\in\mathcal{J}\setminus\{\mathbf{0}\}} f^2_{k,\boldsymbol{\alpha}}\mathbb{E}[H^2_{\boldsymbol{\alpha}}(\boldsymbol{\xi})]}$$

$$\mathcal{I}^*_i = \{\boldsymbol{\alpha}\in\mathbb{N}^s : 0 \le \sum_{j=1}^{s}\alpha_j \le p, \boldsymbol{\alpha}_i \neq 0 \ \wedge \boldsymbol{\alpha}_l \neq 0 \Longleftrightarrow l \neq i, \forall l = 1,\dots,s\}$$

$$\mathbb{E}[H^2_{\boldsymbol{\alpha}}(\boldsymbol{\xi})] = \int H^2_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\mathrm{d}\mathbb{P}(\boldsymbol{\xi}) = \int\cdots\int_s \prod_{j=1}^{s}(H^2_{\alpha,j}(\xi_j))\mathrm{d}\mathbb{P}(\xi_1)\cdots\mathrm{d}\mathbb{P}(\xi_s) = \prod_{j=1}^{s} p_{\alpha,j}!$$

Formulation of using global sensitivity matrix

Global sensitivity matrix for given values of design variables:

$$\widetilde{\boldsymbol{f}}_{\boldsymbol{\xi_d}^{\chi}}(\boldsymbol{\xi_m}, \boldsymbol{\xi_b}) = \widetilde{\boldsymbol{f}}(\boldsymbol{\xi})|_{\boldsymbol{\xi_d}=\boldsymbol{\xi_d}^{\chi}}$$

Complete global sensitivity matrix:

$$\mathbf{S}^*(\boldsymbol{\xi_d}^{\chi}) = \begin{bmatrix} \cdots & S^*_{f_1,\xi_{m_i}}(\boldsymbol{\xi_d}^{\chi}) & \cdots & S^*_{f_1,\xi_{b_i}}(\boldsymbol{\xi_d}^{\chi}) & \cdots \\ \cdots & S^*_{f_2,\xi_{m_i}}(\boldsymbol{\xi_d}^{\chi}) & \cdots & S^*_{f_2,\xi_{b_i}}(\boldsymbol{\xi_d}^{\chi}) & \cdots \\ & \vdots & & \vdots & \end{bmatrix}$$

Global sensitivity to material properties:

$$\mathbf{S}^*_{\boldsymbol{m}}(\boldsymbol{\xi_d}^{\chi}) = \begin{bmatrix} \cdots & S^*_{f_1,\xi_{m_i}}(\boldsymbol{\xi_d}^{\chi}) & \cdots \\ \cdots & S^*_{f_2,\xi_{m_i}}(\boldsymbol{\xi_d}^{\chi}) & \cdots \\ & \vdots & \end{bmatrix}$$

Optimality measures:

$$A^*(\mathbf{S}^*_{\boldsymbol{m}}) = \mathrm{trace}\left(\left[(\mathbf{S}^*_{\boldsymbol{m}})^{\mathrm{T}}\,\mathbf{S}^*_{\boldsymbol{m}}\right]^{-1}\right),$$

$$D^*(\mathbf{S}^*_{\boldsymbol{m}}) = \det\left((\mathbf{S}^*_{\boldsymbol{m}})^{\mathrm{T}}\,\mathbf{S}^*_{\boldsymbol{m}}\right),$$

$$E^*(\mathbf{S}^*_{\boldsymbol{m}}) = \mathrm{cond}\left((\mathbf{S}^*_{\boldsymbol{m}})^{\mathrm{T}}\,\mathbf{S}^*_{\boldsymbol{m}}\right),$$

Illustrative example – physical problem

Linear non-stationary heat problem

Energy balance equation

$$C\frac{\mathrm{d}\theta}{\mathrm{d}t} = \nabla \cdot [\lambda \nabla \theta],$$

+ Boundary conditions
+ Initial conditions

Discretized system of equations

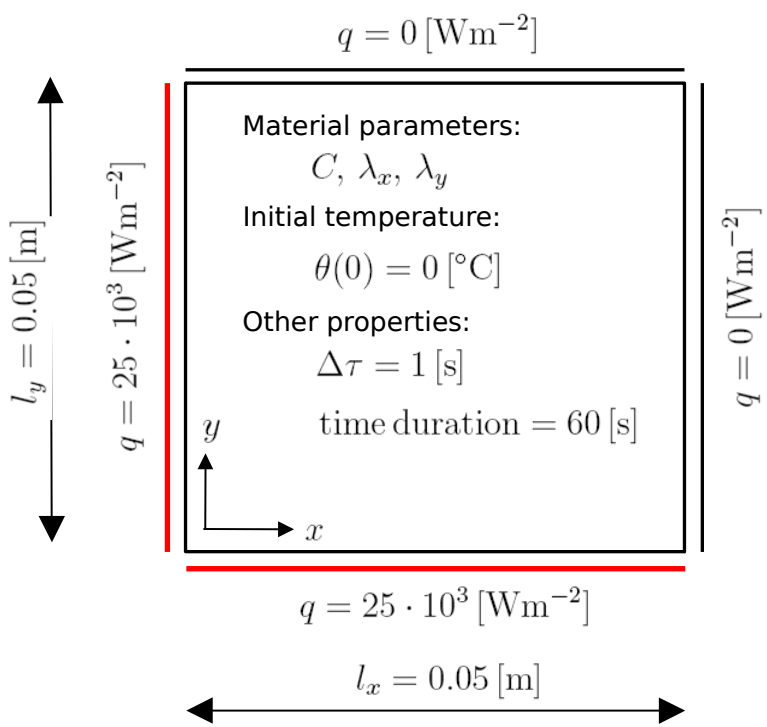
$$\mathbf{C}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \mathbf{K}\boldsymbol{r} = \boldsymbol{q},$$

Solution at time step i+1

$$(\mathbf{C} + \xi\Delta\tau\mathbf{K})\boldsymbol{r}^{i+1} = \xi\Delta\tau\boldsymbol{q} + \mathbf{C}[\boldsymbol{r}^{i} + \Delta\tau(1-\xi)\dot{\boldsymbol{r}}^{i}],$$

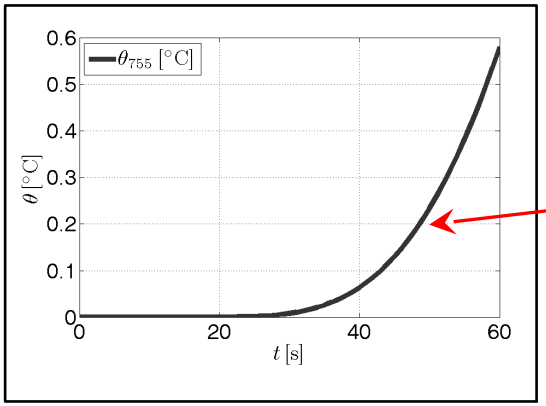
θ	$[°\mathrm{C}]$	temperature
λ	$[\mathrm{Wm^{-1}K^{-1}}]$	thermal conductivity
$C = \rho c$	$[\mathrm{Jm^{-3}K^{-1}}]$	capacity term
c	$[\mathrm{Jkg^{-1}K^{-1}}]$	specific heat capacity
ρ	$[\mathrm{kgm^{-3}}]$	mass density

Initial scheme

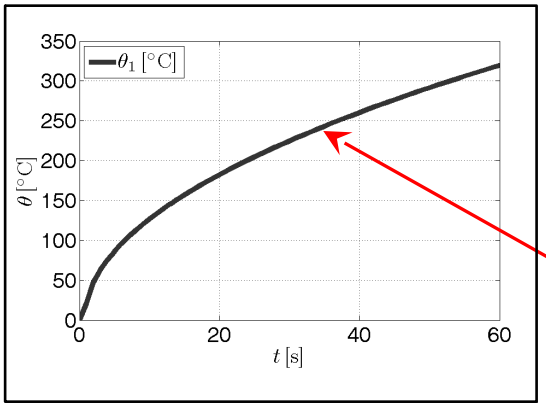


Illustrative example – physical problem

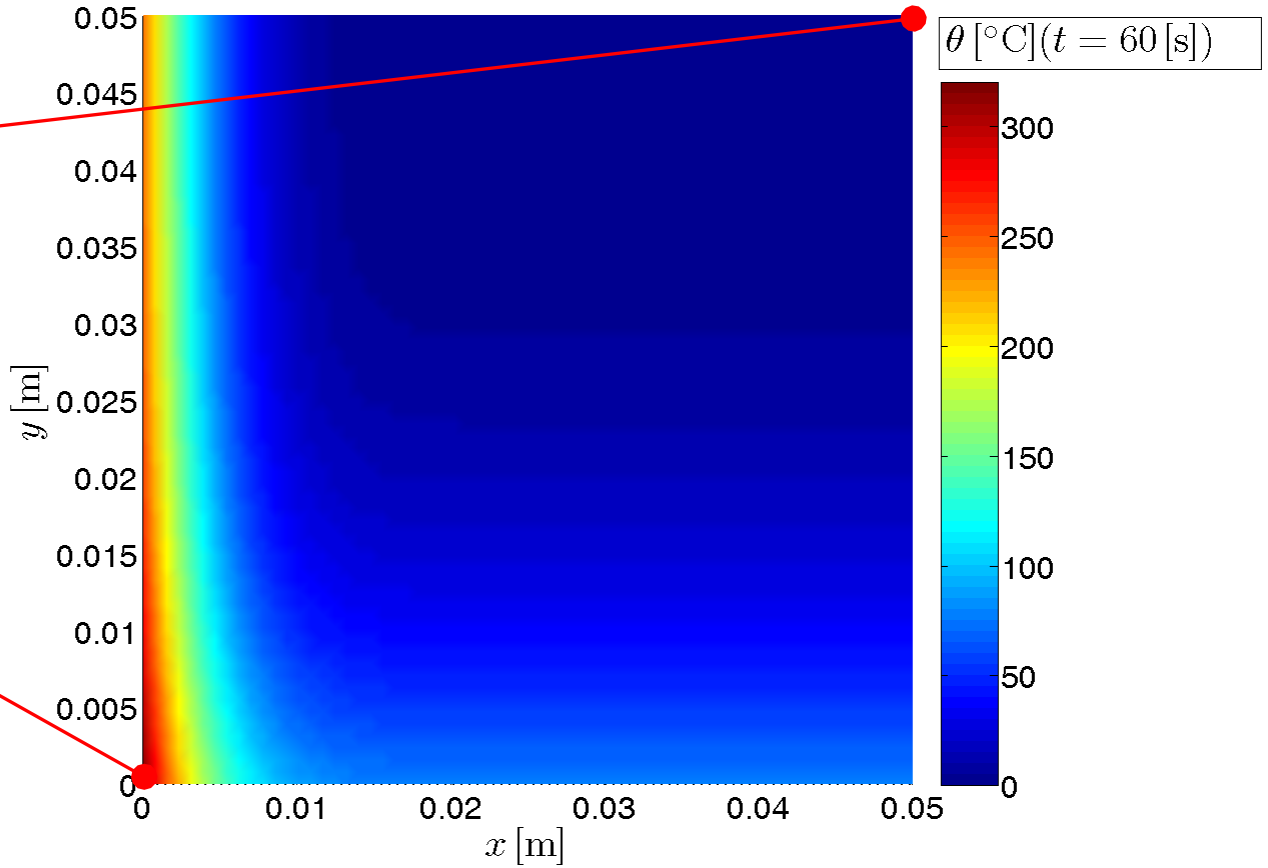
Linear non-stationary heat problem, numerical example



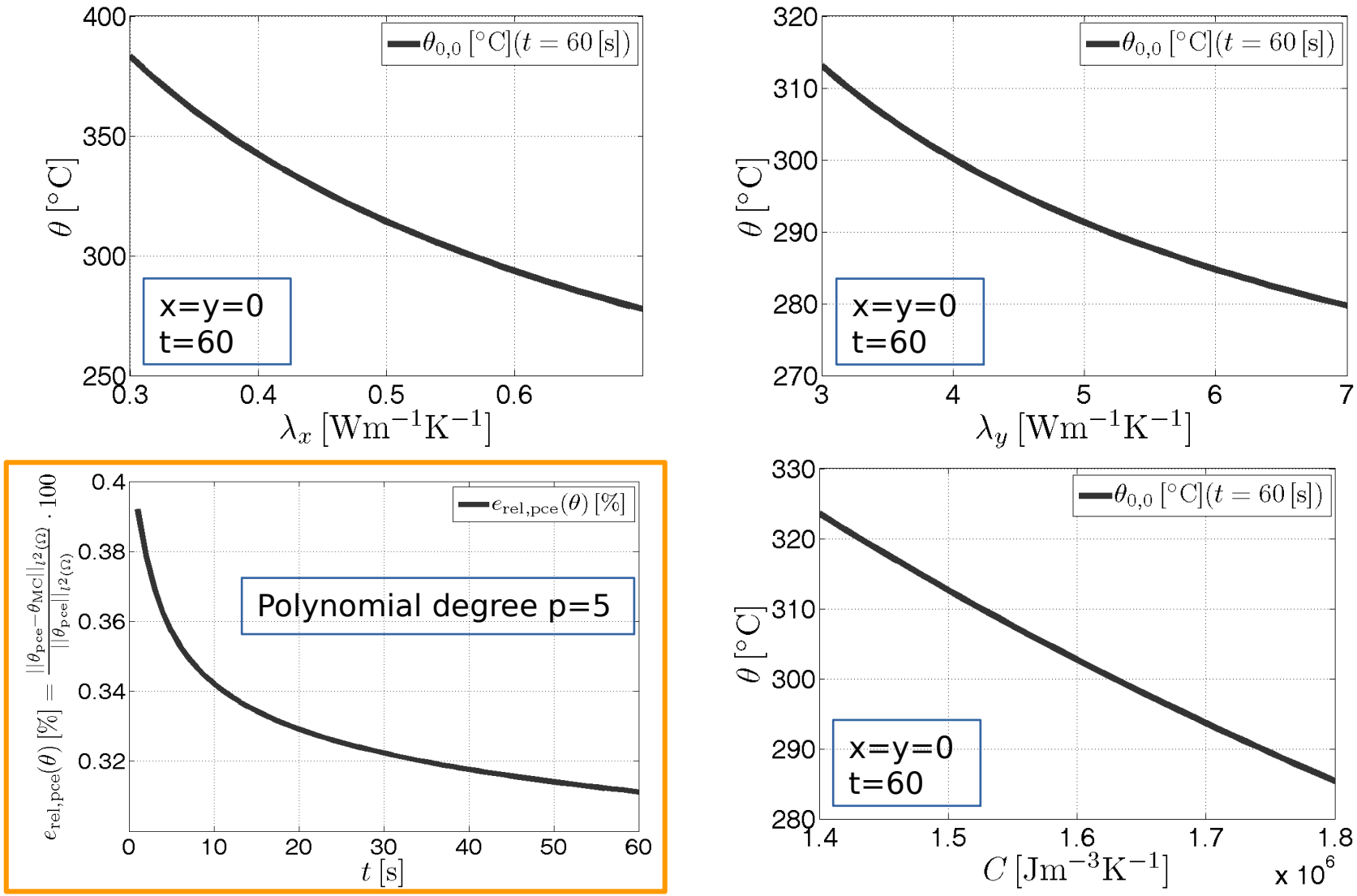
Evolution of temperature



$C = 1.7 \cdot 10^6\,[\mathrm{Jm^{-3}K^{-1}}],\ \lambda_x = 0.6\,[\mathrm{Wm^{-1}K^{-1}}],\ \lambda_y = 4.7\,[\mathrm{Wm^{-1}K^{-1}}]$



Polynomial chaos – error analysis



Experiment design problem definition

Material parameters to be identified:

- $\boldsymbol{m}$ - values of thermal conductivities and heat capacity (3 variables) : $\boldsymbol{m} = (\lambda_x, \lambda_y, C)$

$$\lambda_x \in [0.3; 0.7] \quad \mathrm{Wm^{-1}K^{-1}}$$
$$\lambda_y \in [3.0; 7.0] \quad \mathrm{Wm^{-1}K^{-1}}$$
$$C \in [1.4 \times 10^6; 1.8 \times 10^6] \quad \mathrm{Jm^{-3}K^{-1}}$$

Design variables to be optimised:

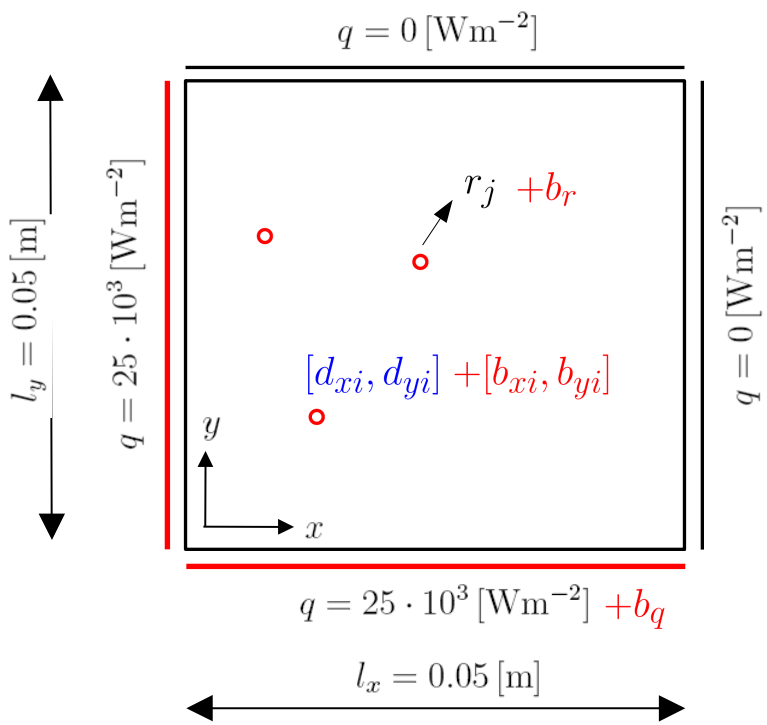
- $\boldsymbol{d}$ - positions of 3 sensors (6 variables) :

$$\boldsymbol{d} = (d_{x1}, d_{y1}, d_{x2}, d_{y2}, d_{x3}, d_{y3})$$

Noise variables to be considered:

- all zero-mean normal variables
- $\boldsymbol{b}_r$ - measurement errors – at 3 sensors at 60 time steps: 180 i.i.d. variables, $\sigma_r = 0.1°\mathrm{C}$
- $\boldsymbol{b}_x$ - inaccurate positions of sensors, 6 i.i.d. var., $\boldsymbol{b}_x = (b_{x1}, b_{y1}, b_{x2}, b_{y2}, b_{x3}, b_{y3})$, $\sigma_{xi} = \sigma_{yi} = 0.5\mathrm{mm}$
- b_q - imperfection in loading $\sigma_q = 50\mathrm{W/m^2}$

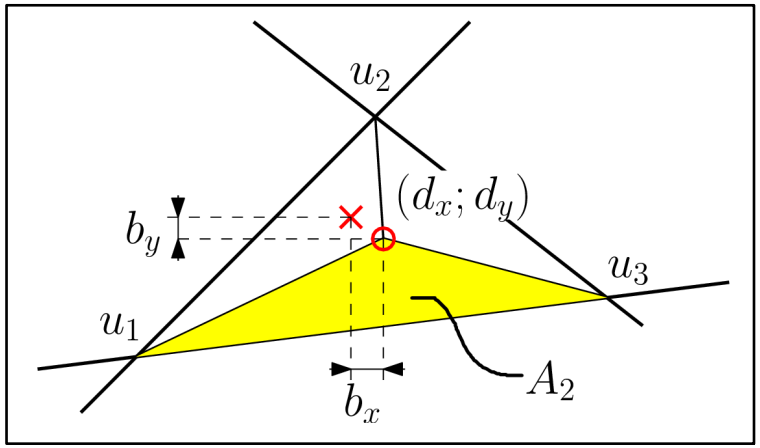
Initial scheme



FEM & Variable sensor positions

- design variables: sensor positions $\boldsymbol{d} \;=\; (d_x, d_y) = \boldsymbol{g}_d(\boldsymbol{\xi}_d)$
- noise variables: error in sensor positions $\boldsymbol{b}_d \;=\; (b_x, b_y) = \boldsymbol{g}_b(\boldsymbol{\xi}_b)$

FE discretisation of model response



$$\begin{aligned}
\mathbf{A}(\boldsymbol{\xi}_m)\boldsymbol{u} &= \boldsymbol{q} \\
\tilde{\boldsymbol{u}}(\boldsymbol{\xi}_m) &= \sum_{\alpha\in\mathcal{I}} \boldsymbol{\beta}_\alpha \psi_\alpha(\boldsymbol{\xi}_m)
\end{aligned}$$

- localisation of k-th element

$$\begin{aligned}
\tilde{\boldsymbol{u}}_k(\boldsymbol{\xi}_m) &= (\tilde{u}_1(\boldsymbol{\xi}_m), \tilde{u}_2(\boldsymbol{\xi}_m), \tilde{u}_3(\boldsymbol{\xi}_m))^{\mathrm{T}} \\
\boldsymbol{N}_k(\boldsymbol{d}(\boldsymbol{\xi}_d)) &= (N_1(\boldsymbol{d}(\boldsymbol{\xi}_d)), N_2(\boldsymbol{d}(\boldsymbol{\xi}_d)), N_3(\boldsymbol{d}(\boldsymbol{\xi}_d))) = \left(\frac{A_1(\boldsymbol{\xi}_d)}{A}, \frac{A_2(\boldsymbol{\xi}_d)}{A}, \frac{A_3(\boldsymbol{\xi}_d)}{A}\right) \\
\boldsymbol{\nabla} &= (\nabla x, \nabla y)^{\mathrm{T}}
\end{aligned}$$

$$\begin{aligned}
f_k(\boldsymbol{\xi}_m, \boldsymbol{\xi}_d, \boldsymbol{\xi}_b) &= \boldsymbol{N}_k(\boldsymbol{\xi}_d)\tilde{\boldsymbol{u}}_k(\boldsymbol{\xi}_m) + \boldsymbol{b}_d(\boldsymbol{\xi}_b)\left[\boldsymbol{\nabla}\boldsymbol{N}_k(\boldsymbol{\xi}_d)\right]\tilde{\boldsymbol{u}}_k(\boldsymbol{\xi}_m) = \left[\boldsymbol{N}_k(\boldsymbol{\xi}_d) + \boldsymbol{b}_d(\boldsymbol{\xi}_b)\left[\boldsymbol{\nabla}\boldsymbol{N}_k(\boldsymbol{\xi}_d)\right]\right]\tilde{\boldsymbol{u}}_k(\boldsymbol{\xi}_m) \\
f_k(\boldsymbol{\xi}_m, \boldsymbol{\xi}_d, \boldsymbol{\xi}_b) &= \sum_{\alpha\in\mathcal{I}} \boldsymbol{\beta}_\alpha \psi_\alpha(\boldsymbol{\xi}_m, \boldsymbol{\xi}_d, \boldsymbol{\xi}_b)
\end{aligned}$$

Experiment design problem definition

Material parameters to be identified:

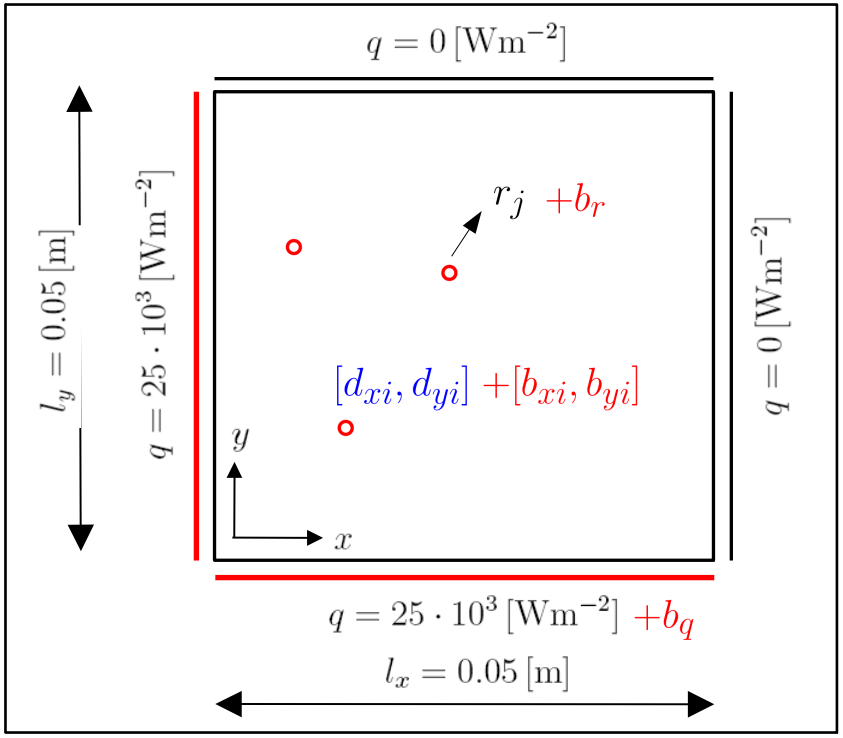
- $\boldsymbol{m}$ - values of thermal conductivities and heat capacity (3 variables) : $\boldsymbol{m} = (\lambda_x, \lambda_y, C)$

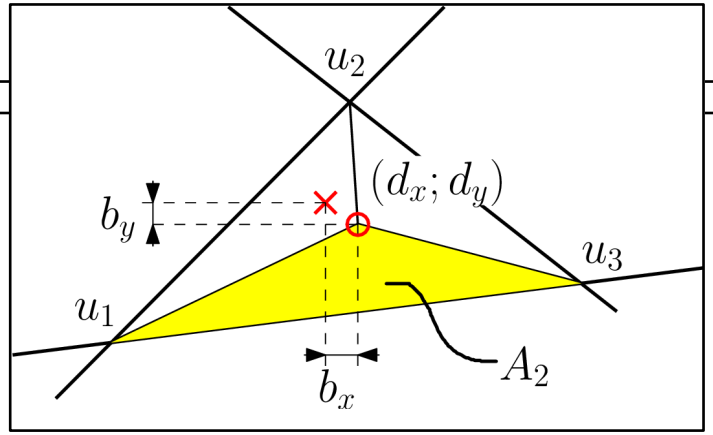
Design variables to be optimised:

- $\boldsymbol{d}$ - positions of 3 sensors (6 variables) :

Noise variables to be considered:

- $\boldsymbol{b}_r$ - measurement errors,
- $\boldsymbol{b}_x$ - inaccurate positions of sensors,
- b_q - imperfection in loading



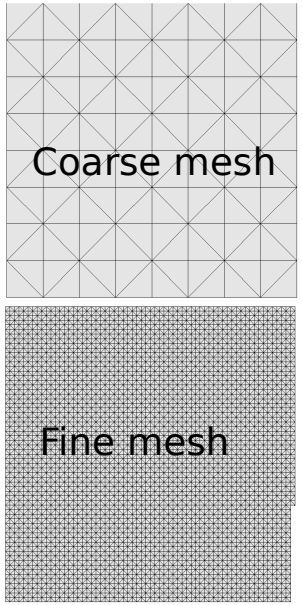


$$\tilde{\boldsymbol{u}}(\cdot) = \sum_{\alpha \in \mathcal{I}} \boldsymbol{\beta}_\alpha \psi_\alpha(\boldsymbol{\xi}_m, \xi_{b_{r_k}})$$

$$f_k(\cdot) = \left[\boldsymbol{N}_k(\boldsymbol{\xi}_d) + \boldsymbol{\xi}_{b_x}\left[\boldsymbol{\nabla}\boldsymbol{N}_k(\boldsymbol{\xi}_d)\right]\right] \tilde{\boldsymbol{u}}_k(\boldsymbol{\xi}_m, \xi_{b_q}) + \xi_{b_{r_k}}$$

$$f_k(\cdot) = \sum_{\alpha \in \mathcal{I}} \boldsymbol{\beta}_\alpha \psi_\alpha(\boldsymbol{\xi}_m, \boldsymbol{\xi}_d, \xi_{b_{r_k}}, \boldsymbol{\xi}_{b_x}, \xi_{b_q})$$

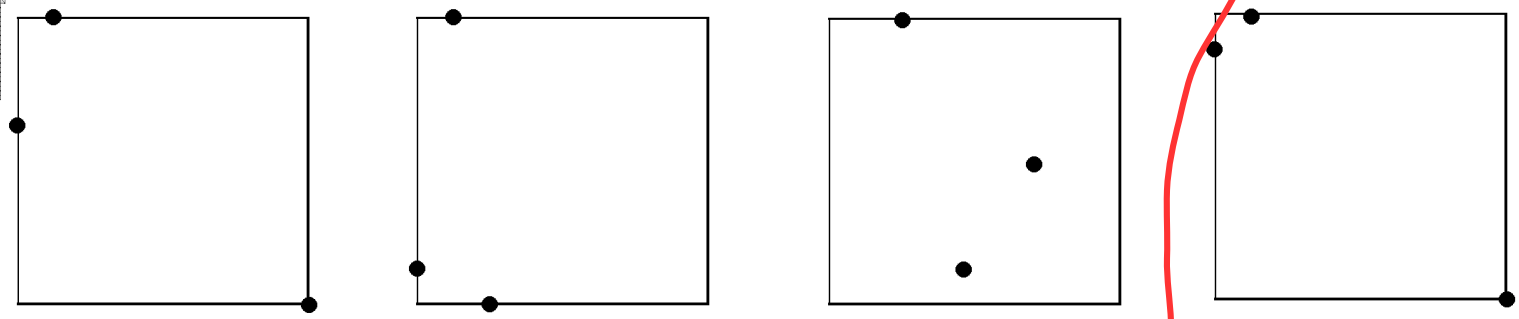
Linearised inverse analysis, global sensitivity



$$\bar{\boldsymbol{m}}(\cdot) = \left(\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}}\mathbf{S}_{\boldsymbol{m}}\right)^{-1}\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}}\left[\boldsymbol{r}(\boldsymbol{d}) - \boldsymbol{f}(\boldsymbol{d},\boldsymbol{b}_d,\boldsymbol{m}) - \mathbf{S}_{\boldsymbol{b}_d} + \mathbf{S}_{\boldsymbol{m}}\bar{\boldsymbol{m}}\right]$$

$$\boxed{\mathrm{cov}(\bar{\boldsymbol{m}}) = \left(\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}}\mathbf{S}_{\boldsymbol{m}}\right)^{-1}\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}}\left[\mathrm{cov}(\boldsymbol{b}_r) + \mathbf{S}_{\boldsymbol{b}_d}\mathrm{cov}(\boldsymbol{b}_d)\mathbf{S}_{\boldsymbol{b}_d}^{\mathrm{T}}\right]\mathbf{S}_{\boldsymbol{m}}\left(\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}}\mathbf{S}_{\boldsymbol{m}}\right)^{-1}}$$

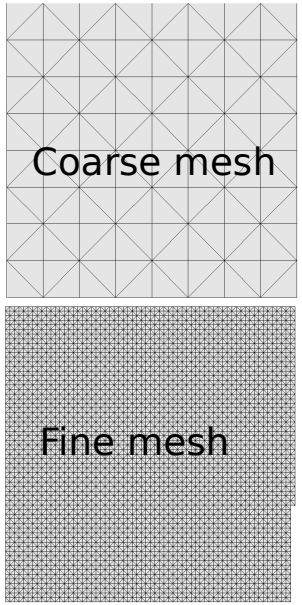
$$\mathrm{cov}(\bar{\boldsymbol{m}}) = \mathrm{diag}\left(\frac{\sigma_{\lambda_x}}{\lambda_x}, \frac{\sigma_{\lambda_y}}{\lambda_y}, \frac{\sigma_C}{C}\right)$$



	A-opt.coarse	A-opt.fine	E-opt.fine	D-opt.fine	
$\sigma_{\lambda_x}/\lambda_x$	20.6% (0.03%)	19.1% (0.03%)	12.1% (0.29%)	20.2%	(0.03%)
$\sigma_{\lambda_y}/\lambda_y$	13.0% (0.05%)	16.2% (0.06%)	5.64% (0.20%)	13.7%	(0.05%)
σ_C/C	9.92% (0.02%)	10.3% (0.02%)	5.10% (0.06%)	10.4%	(0.02%)

Considering only measurement errors: $\mathrm{cov}(\bar{\boldsymbol{m}}) = \sigma_{b_r}^2\left(\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}}\mathbf{S}_{\boldsymbol{m}}\right)^{-1}$

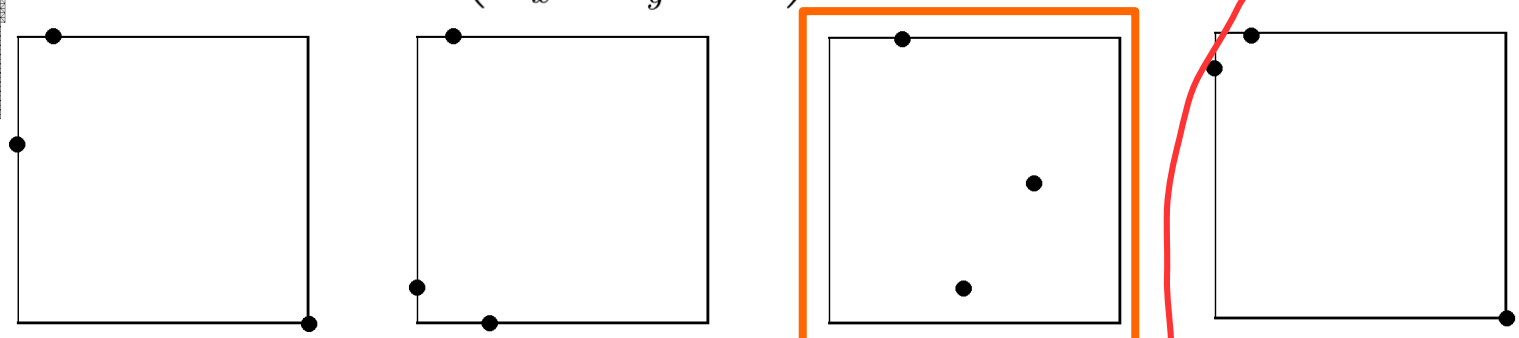
Linearised inverse analysis, global sensitivity



$$\bar{\boldsymbol{m}}(\cdot) = \left(\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}}\mathbf{S}_{\boldsymbol{m}}\right)^{-1}\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}}\left[\boldsymbol{r}(\boldsymbol{d}) - \boldsymbol{f}(\boldsymbol{d},\boldsymbol{b}_d,\boldsymbol{m}) - \mathbf{S}_{\boldsymbol{b}_d} + \mathbf{S}_{\boldsymbol{m}}\bar{\boldsymbol{m}}\right]$$

$$\mathrm{cov}(\bar{\boldsymbol{m}}) = \left(\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}}\mathbf{S}_{\boldsymbol{m}}\right)^{-1}\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}}\left[\mathrm{cov}(\boldsymbol{b}_r) + \mathbf{S}_{\boldsymbol{b}_d}\mathrm{cov}(\boldsymbol{b}_d)\mathbf{S}_{\boldsymbol{b}_d}^{\mathrm{T}}\right]\mathbf{S}_{\boldsymbol{m}}\left(\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}}\mathbf{S}_{\boldsymbol{m}}\right)^{-1}$$

$$\mathrm{cov}(\bar{\boldsymbol{m}}) = \mathrm{diag}\left(\frac{\sigma_{\lambda_x}}{\lambda_x}, \frac{\sigma_{\lambda_y}}{\lambda_y}, \frac{\sigma_C}{C}\right)$$



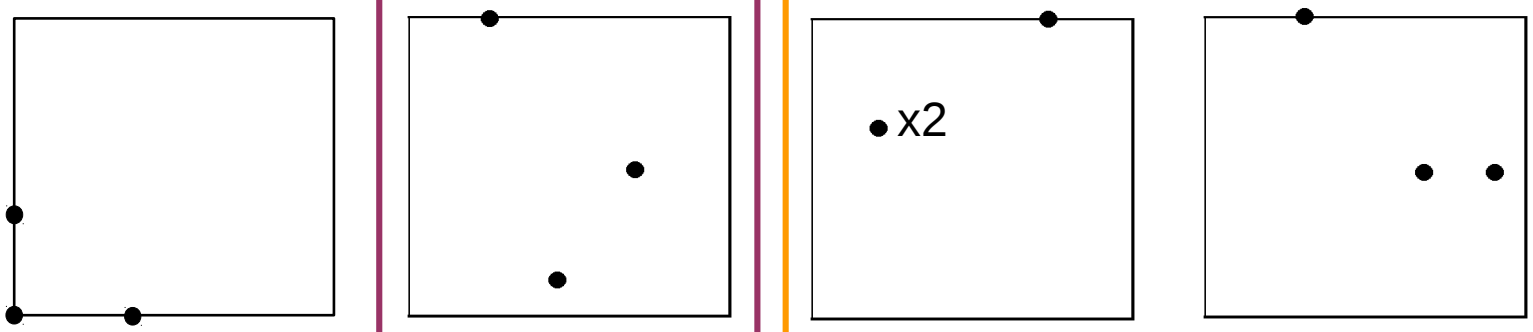
	A-opt.coarse	A-opt.fine	E-opt.fine	D-opt.fine	
$\sigma_{\lambda_x}/\lambda_x$	20.6% (0.03%)	19.1% (0.03%)	12.1% (0.29%)	20.2%	(0.03%)
$\sigma_{\lambda_y}/\lambda_y$	13.0% (0.05%)	16.2% (0.06%)	5.64% (0.20%)	13.7%	(0.05%)
σ_C/C	9.92% (0.02%)	10.3% (0.02%)	5.10% (0.06%)	10.4%	(0.02%)

Considering only measurement errors: $\mathrm{cov}(\bar{\boldsymbol{m}}) = \sigma_{b_r}^2\left(\mathbf{S}_{\boldsymbol{m}}^{\mathrm{T}}\mathbf{S}_{\boldsymbol{m}}\right)^{-1}$

Linearised inverse analysis

Global sensitivity

Local sensitivity



	Sawaf	E-opt.fine	U1-design	U2-design
$\sigma_{\lambda_x}/\lambda_x$	125.2% (0.14%)	12.1% (0.29%)	4.61% (1.58%)	7.96% (1.70%)
$\sigma_{\lambda_y}/\lambda_y$	123.5% (0.11%)	5.64% (0.20%)	7.45% (6.18%)	2.91% (2.34%)
σ_C/C	110.9% (0.13%)	5.10% (0.06%)	2.94% (1.15%)	2.00% (1.26%)

[Sawaf et al., 1995, IJHMT]

[Ruffio, Saury, Petit, 2012, IJHMT]

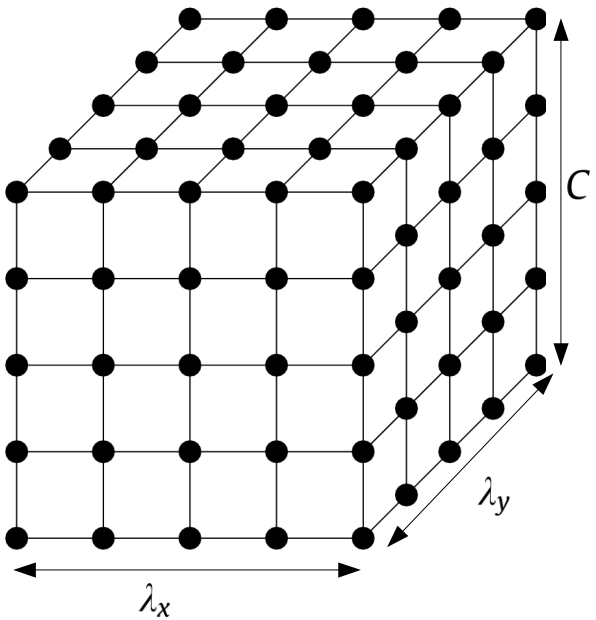
Linearised inverse analysis

Testing of robustness:

- statistics over discretised space of material parameters

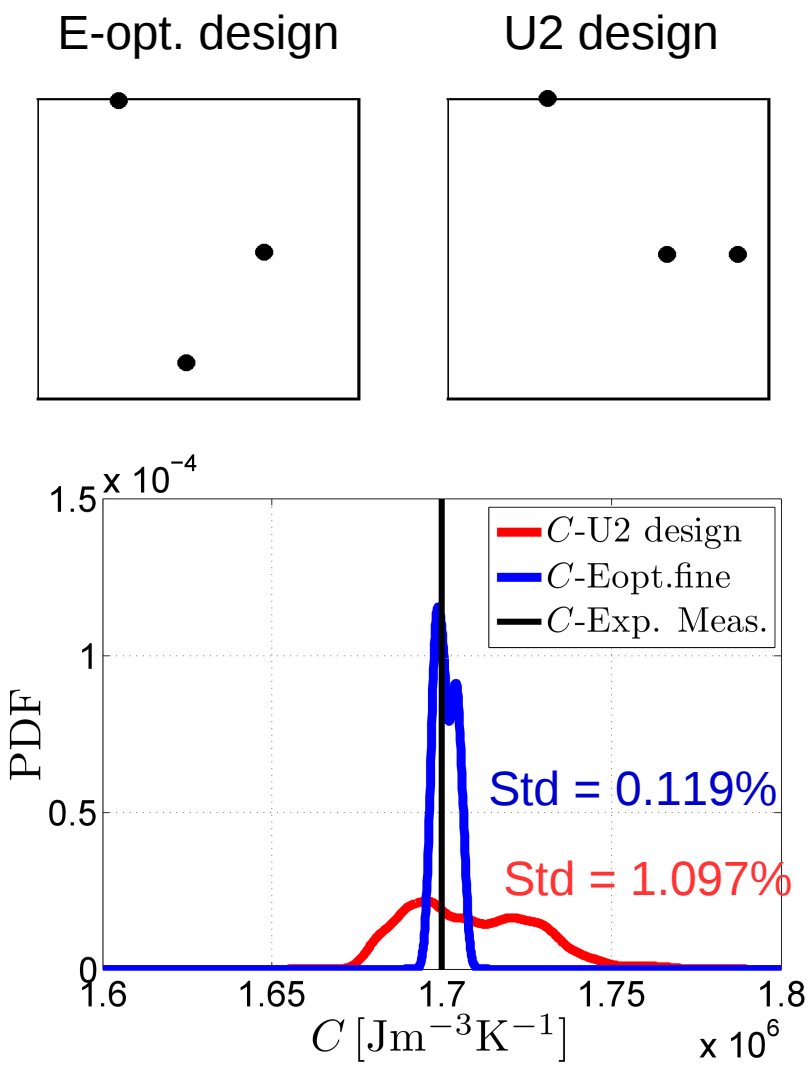
$$\lambda_x \in [0.3; 0.7] \quad \mathrm{Wm^{-1}K^{-1}}$$
$$\lambda_y \in [3.0; 7.0] \quad \mathrm{Wm^{-1}K^{-1}}$$
$$C \in [1.4 \times 10^6; 1.8 \times 10^6] \quad \mathrm{Jm^{-3}K^{-1}}$$

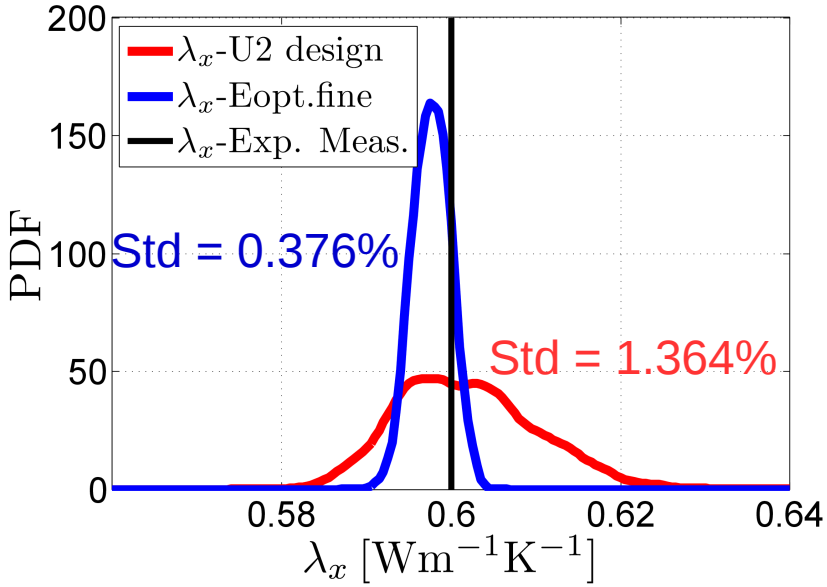
$5 \times 5 \times 5$

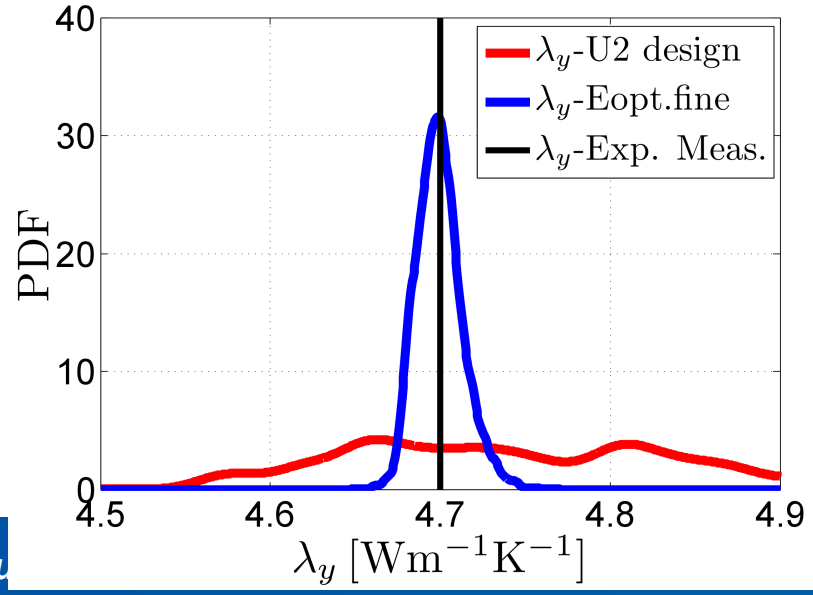


	$\sigma_{\lambda_x}/\lambda_x$	$\sigma_{\lambda_y}/\lambda_y$	σ_C/C
U2-design	7.90-31.9% (0.54-15.2%)	1.73-6.34% (1.06-5.89%)	1.53-4.30% (0.42-3.83%)
A-opt.coarse	18.8-26.7% (0.02-0.03%)	10.2-18.6% (0.04-0.06%)	8.33-14.9% (0.02-0.03%)
A-opt.fine	17.6-22.0% (0.02-0.04%)	15.3-24.2% (0.04-0.07%)	8.63-17.3% (0.02-0.03%)
E-opt.fine	10.6-14.4% (0.17-1.80%)	2.47-6.16% (0.15-0.28%)	3.26-7.18% (0.04-0.07%)
D-opt.fine	18.5-25.2% (0.02-0.03%)	10.5-20.2% (0.04-0.07%)	8.55-16.4% (0.02-0.03%)

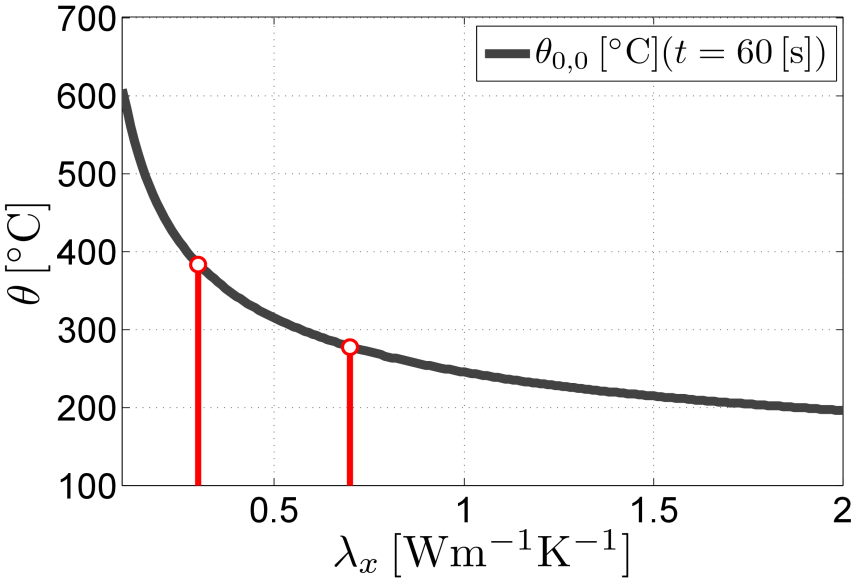
MCMC-based Bayesian inference

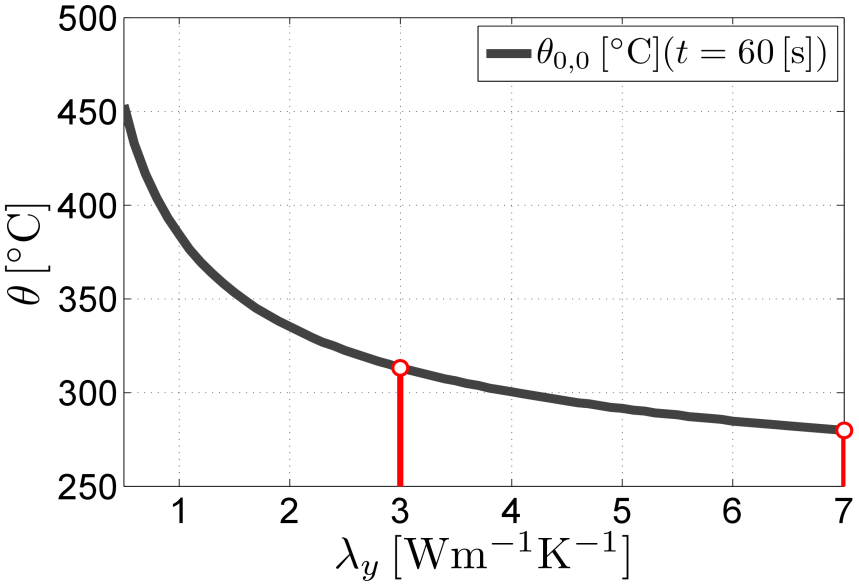


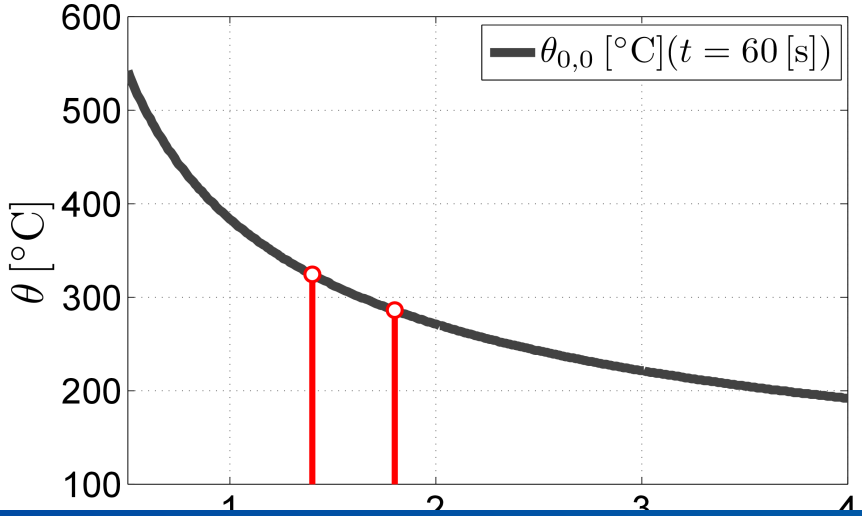




Wider bounds







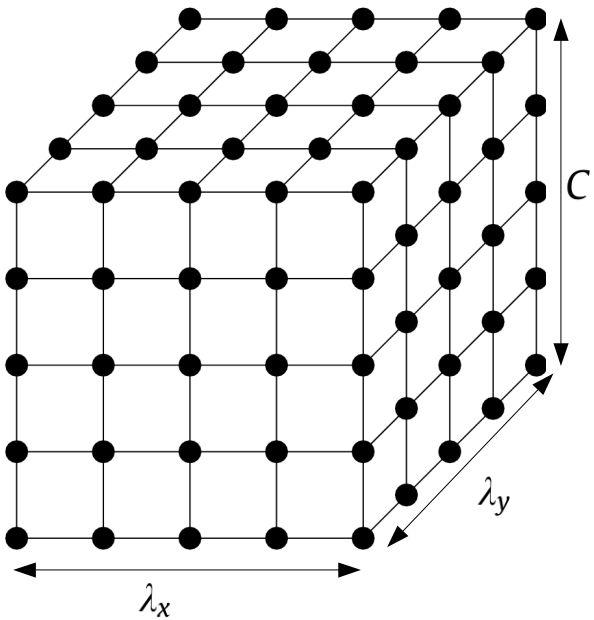
Linearised inverse analysis

Testing of robustness:

- statistics over discretised space of material parameters

$$\lambda_x \in [0.3; 0.7] \quad \mathrm{Wm^{-1}K^{-1}}$$
$$\lambda_y \in [3.0; 7.0] \quad \mathrm{Wm^{-1}K^{-1}}$$
$$C \in [1.4 \times 10^6; 1.8 \times 10^6] \quad \mathrm{Jm^{-3}K^{-1}}$$

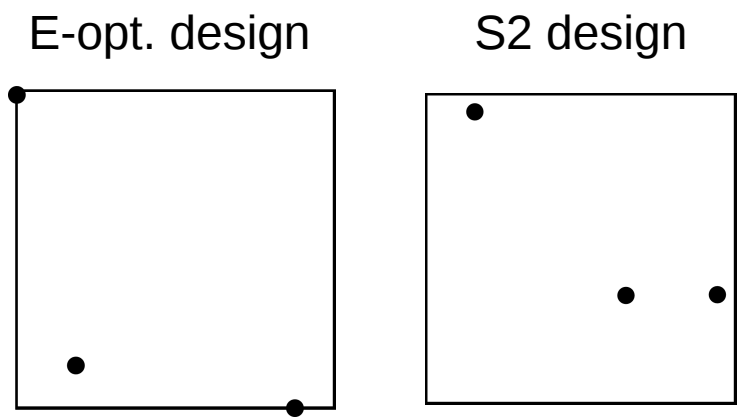
$5 \times 5 \times 5$

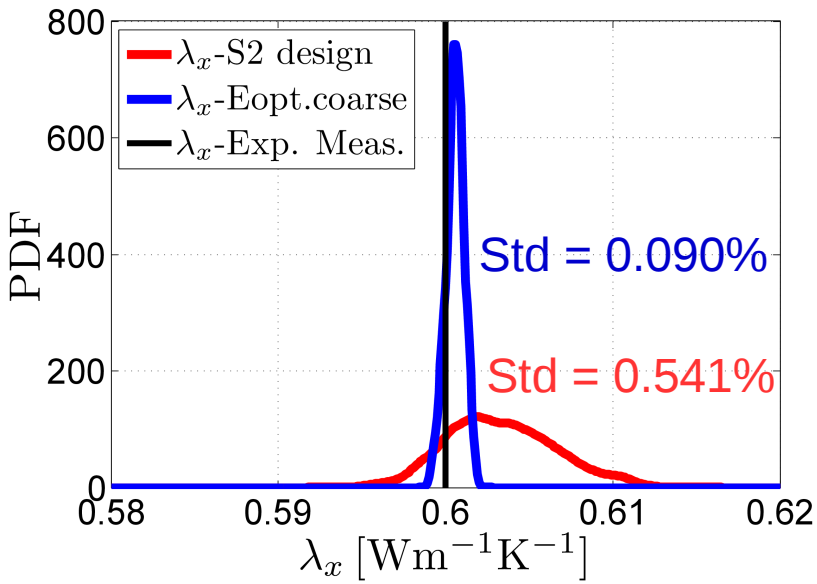


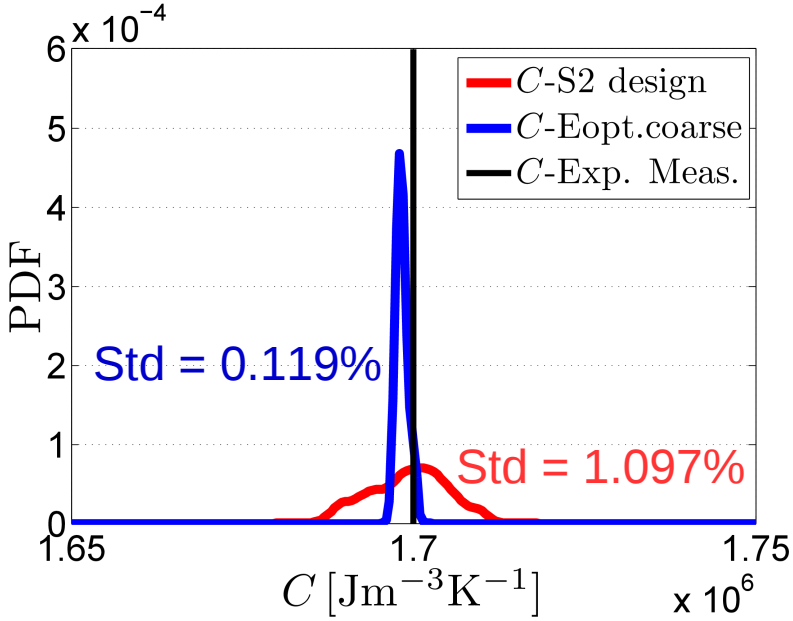
	$\sigma_{\lambda_x}/\lambda_x$	$\sigma_{\lambda_y}/\lambda_y$	σ_C/C
S2	4.11-1664% (0.05-1656%)	1.82-1521% (0.06-1522%)	1.86-12.89% (0.01-12.37%)
E-opt.coarse	9.36-46.31% (0.008-0.11%)	6.56-35.14% (0.01-0.13%)	4.49-24.16% (0.005-0.09%)

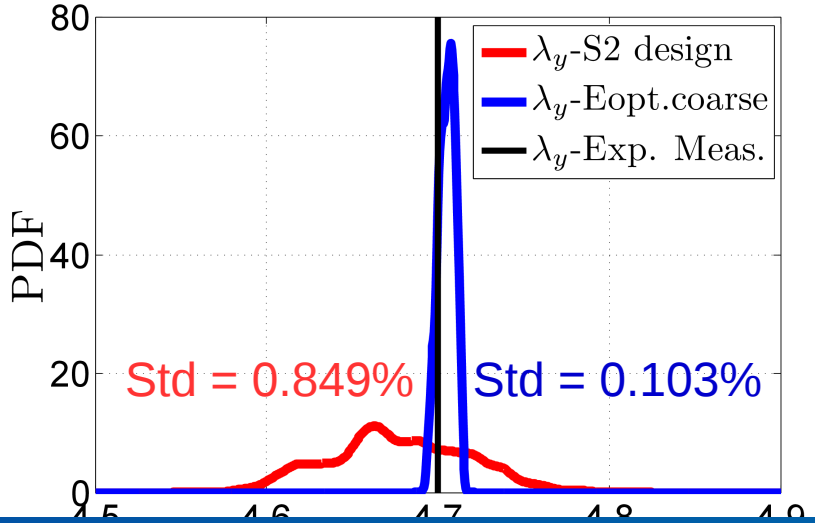
	$\sigma_{\lambda_x}/\lambda_x,\ \lambda_x = 0.6$	$\sigma_{\lambda_y}/\lambda_y,\ \lambda_y = 4.7$	$\sigma_C/C,\ C = 1.7 \cdot 10^6$
S2	14.60% (0.55%)	7.01% (0.76%)	2.51% (0.28%)
E-opt.coarse	20.48% (0.033%)	12.66% (0.059%)	7.71% (0.0274%)

MCMC-based Bayesian inference









Conclusions

Proposed method:

- Global sensitivity matrix – no need for good guess about parameters to be identified, only wide intervals are sufficient
- Better suited for cases, where response is nonlinear w.r.t. material properties
- Involve two approximations: FEM & PCE & total Sobol
  - Cause inaccuracy
  - Decrease computational demands – allow to solve more complex tasks

Future work:

- Investigation of optimality criteria – multi-objective optimisation, minimisation of correlation among the identified parameters etc.

Thank you for your attention

The financial support of this research by the Czech Science Foundation, project No. P15-07299S, is gratefully acknowledged.